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A THEOREM ON THE PLETHYSM OF S-FUNCTIONS

By M. J. NEWELL (*Galway*)

[Received 2 March 1950]

1. THE plethysm of S-functions as defined by Littlewood (**1**) is extremely useful in invariant theory, and the importance of its applications has been demonstrated by its originator (**2**, **3**, etc.). The actual calculation of $\{\lambda\} \otimes \{\mu\}$ is, however, a rather tedious process.

This paper contains the proof of a theorem which is a generalization of one used by Littlewood, and it is shown that it simplifies very much the calculation of $\{\lambda\} \otimes \{\mu\}$. To illústrate the applications some results (not given by Littlewood) on the concomitants of the quintic in any number of variables are given, as well as some plethysms relating to the alternating concomitant types of a set of cubics and quartics.

2. It is known [(**2**) 349] that, if

$$\{\lambda\} \otimes \{n\} = \sum \{\nu\},$$

then

$$\sum g_{1\xi\nu}\{\xi\} = \{\lambda\} \otimes \{n-1\}[g_{1\mu\lambda}\{\mu\}],$$

where g_{rst} is defined from the multiplication of S-functions by means of $\{r\}\{s\} = g_{rst}\{t\}$.

In particular, if $\{\lambda\}$ has only one part m and

$$\{m\} \otimes \{n\} = \sum \{\nu\},$$

then

$$\sum g_{1\xi\nu}\{\xi\} = [\{m\} \otimes \{n-1\}]\{m-1\}.$$

Since $r+s$ is the maximum number of parts for an S-function appearing in the product of two S-functions having r and s parts respectively, it is seen, by induction on n, that any S-function appearing in $\{m\} \otimes \{n\}$ has at most n parts. Furthermore, if $\{\mu\}$ is any partition of n, the same is true for $\{m\} \otimes \{\mu\}$ as can be seen at once from the Jacobi–Trudi determinant associated with $\{\mu\}$.

THEOREM I. *If $\{m\} \otimes \{n\} = \sum \{\nu\}$, where m and n are integers, then for any integer $k \leqslant n$,*

$$\sum g_{(1^k)\xi\nu}\{\nu\} = [\{m-1\} \otimes \{1^k\}][\{m\} \otimes \{n-k\}].$$

Dividing the indeterminates in $\{m\}$ into two sets, of which α will denote the first set and β will denote the second set, we get

$$\{m\} = h_m(\alpha)+h_{m-1}(\alpha)h_1(\beta)+h_{m-2}(\alpha)h_2(\beta)+...+h_m(\beta).$$

In the expansion for $1/\prod(1-xz)$ where z ranges over the terms of $\{m\}$

and in which the coefficient of x^n is $\{m\} \otimes \{n\}$, we can group the factors $(1-xz)$ into $m+1$ sets as indicated by the above equation and obtain the expansion as a product of $m+1$ series,

$$\begin{array}{l} a_0^0+a_1^0x+a_2^0x^2+\ldots+a_l^0x^l+\ldots, \\ a_0^1+a_1^1x+a_2^1x^2+\ldots+a_l^1x^l+\ldots, \\ a_0^2+a_1^2x+a_2^2x^2+\ldots+a_l^2x^l+\ldots, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_0^m+a_1^mx+a_2^mx^2+\ldots+a_l^mx^l+\ldots. \end{array}$$

Since a_l^t is the lth complete symmetric function of the terms got by multiplying any term of $h_{m-t}(\alpha)$ by any term of $h_t(\beta)$, it is equal [(4) 115] to the sum of the products got by multiplying each S-function of weight l in the first set by the corresponding S-function of the second set;

$$a_l^t = \sum_{(\lambda)} [\{m-t\}_\alpha \otimes \{\lambda\}][\{t\}_\beta \otimes \{\lambda\}]. \tag{2.1}$$

The S-functions of β occurring in a_l^t have, at most, l parts, and a_l^t is of weight lt in the β's.

In finding $\{m\} \otimes \{n\}$, which is the coefficient of x^n in the product of the $m+1$ series, let us restrict ourselves to that part of it which is of weight k in the β's. If $a_{p_0}^0 a_{p_1}^1 \ldots a_{p_k}^k$ is such a term, then

$$p_0+p_1+p_2+\ldots+p_k = n,$$

$$p_1+2p_2+\ldots+kp_k = k.$$

Then, by subtraction, $p_0 \geqslant n-k$, and, when $p_0 > n-k$, the maximum number of parts in the S-functions of β which occur in this product (viz. $p_1+p_2+\ldots+p_k$) is less than k. Accordingly $a_{n-k}^0 a_k^1$ is the only term containing $\{1^k\}_\beta$ and the coefficient of $\{1^k\}_\beta$ in it is seen from (2.1) to be

$$a_{n-k}^0[\{m-1\}_\alpha \otimes \{1^k\}].$$

Hence the coefficient of $\{1^k\}_\beta$ in $\{m\} \otimes \{n\}$ is

$$[\{m\}_\alpha \otimes \{n-k\}][\{m-1\}_\alpha \otimes \{1^k\}].$$

But, if $\{m\} \otimes \{n\} = \sum \{\nu\}$, the coefficient of $\{1^k\}_\beta$ is known [(5) 4] to be

$$\sum g_{(1^k)\xi\nu}\{\xi\}_\alpha,$$

whence by comparison of the two forms we get the theorem as stated.

If in the preceding proof the expansion for $1/\Pi(1-xz)$ is replaced by the expansion for $\Pi(1-xz)$, which will entail

$$a_l^t = \sum_{(\lambda)} [\{m-t\}_\alpha \otimes \{\lambda\}][\{t\}_\beta \otimes \{\tilde{\lambda}\}],$$

where $\{\tilde{\lambda}\}$ denotes the conjugate partition of $\{\lambda\}$, we deduce the associated result

THEOREM I A. *If* $\{m\} \otimes \{1^n\} = \sum \{\nu\}$, *then for any integer* $k \leqslant n$,

$$\sum g_{(1^k)\xi\nu}\{\xi\} = [\{m-1\} \otimes \{k\}][\{m\} \otimes \{1^{n-k}\}].$$

3. Applications of the theorem

Since $\sum g_{(1^n)\xi\nu}\{\xi\}$ is the result of diminishing by unity each element in the n-element S-functions appearing in $\{m\} \otimes \{n\}$, it follows that those S-functions are determined uniquely by $\{m-1\} \otimes \{1^n\}$; e.g., knowing that

$$\{4\} \otimes \{1^3\} = \{10, 1^2\}+\{93\}+\{831\}+\{75\}+\{741\}+\{63^2\}+\{5^2 2\},$$

we deduce that the set of 3-element S-functions in $\{5\} \otimes \{3\}$ is

$$\{11, 22\}+\{10, 41\}+\{942\}+\{861\}+\{852\}+\{74^2\}+\{6^2 3\}.$$

Again, since $\sum g_{(1^{n-1})\xi\nu}\{\xi\}$ is the result of diminishing by unity each element in the $(n-1)$-element S-functions, together with some S-functions arising from the already determined n-element partitions, the $(n-1)$-element S-functions are uniquely determined by

$$[\{m-1\} \times \{1^{n-1}\}]\{m\}.$$

Thus a general recurrence method is available.

Thus for $\{5\} \otimes \{3\}$ the two-element partitions are got from

$$[\{4\} \otimes \{1^2\}]\{5\}-[\{4\} \otimes \{1^3\}]\{1\},$$

and, since we need only consider S-functions of two parts or less, the terms $\{93\}$ and $\{75\}$ only are taken from $\{4\} \otimes \{1^3\}$.

The expression reads

$$[\{71\}+\{53\}]\{5\}-[\{93\}+\{75\}]\{1\},$$

which in two variables is

$$\{12, 1\}+\{11, 2\}+\{10, 3\}+\{94\}+\{85\}.$$

Hence the two-element S-functions of $\{5\} \otimes \{3\}$ are

$$\{13, 2\}+\{12, 3\}+\{11, 4\}+\{10, 5\}+\{9, 6\}.$$

Since $\{5\} \otimes \{3\}$ clearly contains $\{15\}$, we have the result,

$$\{5\} \otimes \{3\} = \{11, 22\}+\{10, 41\}+\{942\}+\{861\}+\{852\}+\{74^2\}+\{6^2 3\}+ \\ +\{13, 2\}+\{12, 3\}+\{11, 4\}+\{10, 5\}+\{9, 6\}+\{15\}.$$

4. Other expansions

Littlewood (**2**) has given the expansions of $\{2\} \otimes \{n\}$, $\{3\} \otimes \{n\}$, $\{4\} \otimes \{n\}$ $(n = 2, 3, 4, 5)$, and also for $\{4\} \otimes \{1^n\}$ $(n = 2, 3)$.

From the expansions for $\{4\} \otimes \{n\}$ we can read off (by means of Theorem I)

$$\{3\} \otimes \{1^2\} = \{51\}+\{33\}, \qquad \{3\} \otimes \{1^3\} = \{71^2\}+\{63\}+\{531\}+\{3^3\},$$

$$\{3\} \otimes \{1^4\} = \{91^3\}+\{831\}+\{731^2\}+\{741\}+\{6^2\}+\{642\}+ +\{63^2\}+\{5^21^2\}+\{53^21\}+\{3^4\},$$

$$\{3\} \otimes \{1^5\} = \{11, 1^4\}+\{10, 31^2\}+\{942\}+\{941^2\}+\{931^3\}+\{861\}+ +\{851^2\}+\{843\}+\{8421\}+\{83^21\}+\{73^21^2\}+ +\{7521\}+\{751^3\}+\{7431\}+\{74^2\}+\{762\}+\{6^23\}+ +\{6432\}+\{63^3\}+\{6531\}+\{5^231^2\}+\{53^31\}+\{3^5\}.$$

Similar results for the quadratic can be read off from the known expansions for $\{3\} \otimes \{n\}$.

5. Applications of Theorem I A

Theorem I A may be used similarly to evaluate $\{m\} \otimes \{1^n\}$.

Example 1: $\{4\} \otimes \{1^4\}$.

The 4-element partitions can be read off at once from $\{3\} \otimes \{4\}$. They are

$$\{13, 1^3\}+\{11, 31^2\}+\{10, 41^2\}+\{951^2\}+\{93^21\}+\{8521\}+ +\{8431\}+\{7^21^2\}+\{7531\}+\{73^3\}+\{6532\}+\{5^31\}.$$

Each 3-element S-function corresponds to a term of

$$[\{3\} \otimes \{3\}]\{4\}-[\{3\} \otimes \{4\}]\{1\},$$

where in this expression we neglect partitions having more than three parts. Hence we must find

$$[\{9\}+\{72\}+\{63\}+\{52^2\}+\{4^21\}]\{4\}-[\{12\}+\{10, 2\}+ +\{93\}+\{84\}+\{82^2\}+\{741\}+\{732\}+\{6^2\}+\{642\}+\{4^3\}]\{1\}.$$

This can be simplified by pairing terms that yield products which agree in the first two or three terms: e.g.

$$\{9\}\{4\}-\{12\}\{1\} = \{11, 2\}+\{10, 3\}+\{9, 4\},$$

$$\{52^2\}\{4\}-\{82^2\}\{1\} = \{742\}+\{652\},$$

$$\{4^21\}\{4\}-\{741\}\{1\} = \{643\}+\{54^2\}-\{751\}.$$

After these simplifications and some obvious cancelling we arrive at

$$\{11, 2\}+\{10, 3\}+\{94\}+\{931\}+\{92^2\}+\{85\}+\{841\}+ +\{832\}+\{76\}+\{751\}+\{742\}+\{652\}+\{643\}. \tag{a}$$

Hence the 3-element S-functions in $\{4\} \otimes \{1^4\}$ are

$$\{12,31\}+\{11,41\}+\{10,51\}+\{10,42\}+\{10,3^2\}+\{961\}+\{952\}+ \\ +\{943\}+\{871\}+\{862\}+\{853\}+\{763\}+\{754\}.$$

The 2-element S-functions correspond to the partitions of not more than two parts which appear in $[\{3\} \otimes \{2\}][\{4\} \otimes \{1^2\}]$ after having rejected those arising from the 3-element and 4-element partitions in $\{4\} \otimes \{1^4\}$. These partitions to be rejected are clearly the terms in the product of the expression (a) by $\{1\}$ together with those got by deleting the last two parts in any 4-element partition of $\{4\} \otimes \{1^4\}$ which ends in (1^2). Accordingly we must find

$$[\{6\}+\{42\}][\{71\}+\{53\}]-[\{11,2\}+\{10,3\}+\{94\}+\{85\}+\{76\}]\{1\}- \\ -[\{13,1\}+\{11,3\}+\{10,4\}+\{95\}+\{7^2\}],$$

looking only to partitions into not more than two parts.

This is easily seen to be $\{95\}$, so that the only 2-element partition of $\{4\} \otimes \{1^4\}$ is $\{10,6\}$.

Collecting results we have

$$\{4\} \otimes \{1^4\} = \{13,1^3\}+\{12,31\}+\{11,41\}+\{11,31^2\}+\{10,51\}+ \\ +\{10,42\}+\{10,41^2\}+\{10,6\}+\{10,3^2\}+\{961\}+ \\ +\{952\}+\{951^2\}+\{943\}+\{93^21\}+\{871\}+\{862\}+ \\ +\{853\}+\{8521\}+\{8431\}+\{7^21^2\}+\{763\}+\{754\}+ \\ +\{7531\}+\{73^3\}+\{6532\}+\{5^31\}.$$

Example 2: $\{5\} \otimes \{4\}$.

The 4-element S-functions can be written down immediately from the preceding result. The 3-element S-functions depend on

$$[\{4\} \otimes \{1^3\}]\{5\}-[\{4\} \otimes \{1^4\}]\{1\},$$

and will be found to be

$$\{16,2^2\}+\{15,41\}+\{15,32\}+\{14,51\}+2\{14,42\}+2\{13,61\}+ \\ +2\{13,52\}+\{12,71\}+3\{12,62\}+\{12,53\}+2\{12,4^2\}+\{11,81\}+ \\ +3\{11,72\}+2\{11,63\}+\{11,54\}+\{10,91\}+2\{10,82\}+\{10,73\}+ \\ +3\{10,64\}+\{983\}+\{974\}+\{965\}+\{8^24\}+\{86^2\}.$$

The remaining terms are

$$\{20\}+\{18,2\}+\{17,3\}+2\{16,4\}+\{15,5\}+ \\ +2\{14,6\}+\{13,7\}+2\{12,8\}+\{10^2\}.$$

In virtue of Littlewood's theorem of conjugates, each expansion obtained here yields a complementary result for forms of types $\{1^4\}$, $\{1^5\}$.

6. Ternary forms

When the number of variables in the ground-form is limited (as is usual in applications of the theory) the work is correspondingly simplified. Thus, in finding $\{3\} \otimes \{5\}$ for the ternary case, it is easily seen from the preceding examples that the 3-element S-functions are uniquely determined by

$$[\{2\} \otimes \{1^3\}][\{3\} \otimes \{2\}]-[\{2\} \otimes \{1^4\}]\{3\}\{1\}+[\{2\} \otimes \{1^5\}]\{2\}.$$

Neglecting partitions into more than three parts, this is

$$[\{3^2\}+\{41^2\}][\{6\}+\{42\}]-\{431\}[\{4\}+\{31\}]+\{4^22\}\{2\},$$

which on evaluation turns out to be

$$\{10,1^2\}+\{93\}+\{921\}+2\{831\}+\{75\}+\{741\}+$$
$$+\{732\}+\{651\}+\{63^2\}+\{5^22\}.$$

To find the two-element S-functions we must calculate

$$[\{2\} \otimes \{1^3\}][\{3\} \otimes \{3\}]-[\{93\}+\{75\}]\{1\}$$

for two variables.

This is $$[\{9\}+\{72\}+\{63\}]\{31\}-[\{93\}+\{75\}]\{1\},$$

which is clearly

$$\{12,1\}+\{11,2\}+\{10,3\}+\{94\}+\{85\}.$$

Hence

$$\{3\}\times\{5\} = \{15\}+\{13,2\}+\{12,3\}+\{11,4\}+\{10,5\}+\{96\}+\{11,2^2\}+$$
$$+\{10,41\}+\{10,32\}+2\{942\}+\{861\}+\{852\}+\{843\}+\{762\}+\{74^2\}+\{6^23\}.$$

This agrees with the result as given by Littlewood.

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NOTE ON COHOMOLOGY SYSTEMS

By S. C. CHANG (*Hangchow*) *and* J. H. C. WHITEHEAD (*Oxford*)

[Received 25 March 1950]

1. Introduction

IN (**1**) Chang introduces certain new numerical invariants of a polyhedron P, assuming that $\dim P \leqslant n+2$ and $\pi_r(P) = 0$ $(r = 1,..., n-1)$, where $n > 2$. They are defined in terms of the Steenrod homomorphism, $Sq_{n-2}\colon H^n(2) \to H^{n+2}(2)$, which appears in (**2**). In this note we introduce similar invariants of any finite polyhedron. They are defined in terms of Sq_{n-k}, operating on $H^n(2)$, for any even value of k such that $0 < k \leqslant n$.

We recall from (**3**) the definition of a cohomology spectrum. It consists of a 2-index family of additive, Abelian groups $H^n(m)$ $(n = 0, 1,...)$ related by certain homomorphisms μ, Δ. Here m will take only the values 0, 2, and μ, Δ are of the form

$$H^n \xrightarrow{\mu} H^n(2) \xrightarrow{\Delta} H^{n+1},$$

where $H^r = H^r(0)$. Let $\nu\colon H^n \to H^n$ be given by $\nu u = 2u$ $(u \in H^n)$. Then the sequence of homomorphisms

$$\dots \xrightarrow{\mu} H^{n-1}(2) \xrightarrow{\Delta} H^n \xrightarrow{\nu} H^n \xrightarrow{\mu} H^n(2) \xrightarrow{\Delta} \dots, \tag{1.1}$$

which begins with $\Delta\colon 0 \to H^0$, is exact. Thus†

$$\mu H^n \approx H^n_2, \qquad \Delta H^n(2) = {}_2H^{n+1}.$$

Moreover it is assumed that Δ has a right inverse, $\Delta^*\colon {}_2H^{n+1} \to H^n(2)$. Therefore

$$H^n(2) = \mu H^n + \Delta^*({}_2H^{n+1}) \approx H^n_2 + {}_2H^{n+1},$$

where $+$ indicates direct summation. Whereas μ, Δ are uniquely defined, Δ^* is only defined *modulo* an arbitrary homomorphism ${}_2H^{n+1} \to \mu H^n$. Let a homomorphism

$$\gamma = \gamma^n_k\colon H^n(2) \to H^{n+k}(2) \quad (k \equiv 0, \bmod 2)$$

be defined for every even $k > 0$ and every $n \geqslant k$. The entire family of groups $H^n(m)$ $(m = 0, 2)$, related by the homomorphisms μ, Δ, γ, will be called a (μ, Δ, γ)-*system*. We shall denote it by the single letter H.

† $A_2 = A/2A$, and ${}_2A$ consists of all elements $a \in A$ such that $2a = 0$, where A is any additive, Abelian group.

By a (μ, Δ)-*homomorphism* (*isomorphism*),† $f\colon H \to H'$, into a (μ, Δ, γ)-system H', we mean a family of homomorphisms (isomorphisms), $f\colon H^n(m) \to H'^n(m)$, such that

$$f\mu = \mu f, \qquad f\Delta = \Delta f. \tag{1.2}$$

In any case $f\nu = \nu f$. Therefore f is a homomorphism of the sequence (1.1) into the corresponding sequence for H'. It follows from simplified versions of Lemmas 2, 3 in (**3**) that $f\colon H \approx H'$ if $f\colon H^n \approx H'^n$, for every $n \geqslant 0$, and that any set of isomorphisms, $H^n \approx H'^n$ ($n = 0, 1,...$), can be extended to a (μ, Δ)-isomorphism $H \approx H'$. If a (μ, Δ)-homomorphism also commutes with each γ_k^n, we shall call it a (μ, Δ, γ)-*homomorphism*, or simply a *homomorphism* of H into H'.

Let K be a finite, simplicial complex and let $H^n(m) = H^n(K, I_m)$, where I_m is the group of integers reduced mod m. Let $\mu\colon H^n \to H^n(2)$ be the natural homomorphism and let $\Delta = \frac{1}{2}\delta$. That is to say, if $\delta c = 2c_1$ then $\Delta\bar{c} = \bar{c}_1$, where c, c_1 are co-chains and $\bar{c} \in H^n(2)$, $\bar{c}_1 \in H^{n+1}$ are their cohomology classes. Let γ_k^n be the Steenrod homomorphism

$$Sq_{n-k}\colon H^n(2) \to H^{n+k}(2).$$

It follows from (6.1) in (**2**) that, if $n-i-1 = k \equiv 0 \pmod 2$, then

$$Sq_i = \Delta\gamma_k^n\mu\colon H^n \to H^{n+k+1}. \tag{1.3}$$

Therefore the remaining Steenrod squares, with integral coefficients, are determined by μ, Δ, γ.

The following special case of (1.3) was pointed out to one of us by Steenrod. Let $n > 2$ and let M^{n+2} be the cell-complex

$$M^{n+2} = e^0 \cup e^n \cup e^{n+2},$$

where $e^0 \cup e^n = S^n$, say, is an n-sphere and e^{n+2} is attached to S^n by an essential map $\dot{E}^{n+2} \to S^n$. Since $\pi_{n+1}(S^n)$ is of order 2, it follows that $2a \in j\pi_{n+2}(M^{n+2})$, where a is a generator of $\pi_{n+2}(M^{n+2}, S^n)$ and

$$j\colon \pi_{n+2}(M^{n+2}) \to \pi_{n+2}(M^{n+2}, S^n)$$

is the injection. Therefore there is a map $\phi\colon S^{n+2} \to M^{n+2}$ which is of degree 2 in e^{n+2}. Let $S^{n+2} = \dot{E}^{n+3}$ and let $e^{n+3} = E^{n+3} - S^{n+2}$, where E^{n+3} is an $(n+3)$-element which does not meet M^{n+2}. We attach e^{n+3} to M^{n+2} by means of the map ϕ, thus forming a complex K. Let $c^r \in C^r(K)$ be the co-chain which corresponds to the cell‡ e^r. Then $\delta c^n = 0$, $\delta c^{n+2} = 2c^{n+3}$ and

$$c^n \cup_{n-2} c^n = c^{n+2} \tag{1.4}$$

† An isomorphism, without qualification, will always mean an isomorphism onto.

‡ Cf. § 8 of (**3**).

according to § 20 of (**2**). Let

$$u \in H^n, \qquad v \in H^{n+2}(2), \qquad w \in H^{n+3}$$

be the elements corresponding to c^n, c^{n+2}, c^{n+3}. Then $\Delta v = w$ and it follows from (1.3) and (1.4) that

$$Sq_{n-3}\, u = \Delta\gamma_2^n\, \mu u = \Delta v = w. \tag{1.5}$$

2. The block invariants

Let H be a given (μ, Δ, γ)-system and let n, k be given, with $0 < k \leqslant n$, $k \equiv 0 \pmod 2$. Let each group H^r be finitely generated. Then it is a direct sum of cyclic groups of infinite or prime-power order. We shall only be concerned with those summands whose orders are infinite or powers of 2. Let $u_1,\ldots, u_p$ be generators of these summands of H^n. Let m_i be the order of u_i, with $m_i \leqslant m_{i+1} \leqslant \infty$. Let $v_1,\ldots, v_q$ generate the summands of H^{n+1} whose orders are powers of 2. Let $2\rho_s$ be the order of v_s and now let $\rho_s \geqslant \rho_{s+1}$. Then ${}_2H^{n+1}$ is generated by $\rho_1 v_1,\ldots, \rho_q v_q$. Let Δ^* be a right inverse of Δ: $H^n(2) \to H^{n+1}$. Then $H^n(2)$ is a mod 2 module, with a basis consisting of

$$x_i = \mu u_i, \qquad y_s = \Delta^* \rho_s v_s \quad (i = 1,\ldots,p;\ s = 1,\ldots,q). \tag{2.1}$$

We shall describe $\{x_i, y_s\}$ as a (μ, Δ)-*basis* of $H^n(2)$.

Let ϕ: $H^n(2) \approx H^n(2)$ be an automorphism which is given by

$$\phi x_i = \sum_j p_{ij} x_j + \sum_s p'_{is} y_s,$$
$$\phi y_r = \sum_j q_{rj} x_j + \sum_s q'_{rs} y_s,$$

where p_{ij}, etc., are integers reduced mod 2. Then the combined proofs of Lemmas 3 and 4 in (**1**) show† that ϕ can be extended to a (μ, Δ)-automorphism of H if and only if

$$\left.\begin{aligned} p_{ij} &= 0 \quad \text{if} \quad m_i < m_j \\ p'_{is} &= 0 \\ q'_{rs} &= 0 \quad \text{if} \quad \rho_r > \rho_s \end{aligned}\right\}. \tag{2.2}$$

Notice that $i < j$ if $m_i < m_j$ and $r < s$ if $\rho_r > \rho_s$, since $m_i \leqslant m_{i+1}$, $\rho_r \geqslant \rho_{r+1}$.

Similar observations apply to automorphisms of $H^{n+k}(2)$: only we order the elements $u'_i \in H^{n+k}$, $v'_s \in H^{n+k+1}$, which are analogous to u_i, v_s so that $m'_i \geqslant m'_{i+1}$ and $\rho'_s \leqslant \rho'_{s+1}$, where m'_i, $2\rho'_s$ are the orders of u'_i, v'_s.

Let $\mathfrak{g} = \mathfrak{g}_k^n$ be the mod 2 matrix which corresponds to the homomorphism $\gamma = \gamma_k^n$, when γ is expressed in terms of the basis $\{x_i, y_s\}$, for $H^n(2)$, and the basis for $H^{n+k}(2)$, which is defined in the same way but

† See also Lemma 3 in (**3**).

is written in the reverse order. Our *block invariants* $\tau^{\alpha}_{\beta} = \tau^{n\alpha}_{k\beta}$ are now defined in the same way as the secondary torsions in (**1**). They are invariants of the matrix $\mathbf{g}$ under transformations of the form $\mathbf{g} \to \mathbf{f}^{-1}\mathbf{g}\mathbf{f}'$, where $\mathbf{f}$, $\mathbf{f}'$ are the matrices of automorphisms†

$$\phi: H^n(2) \approx H^n(2), \qquad \phi': H^{n+k}(2) \approx H^{n+k}(2)$$

which belong to an arbitrary (μ, Δ)-automorphism of H.

We recall the definition of τ^{α}_{β}. Let $n_1, n_2,...$ be those of $m_1,..., m_p$ which are distinct and let $\sigma_1, \sigma_2,...$ be those of $\rho_1,..., \rho_q$ which are distinct, with $n_i < n_{i+1} \leqslant \infty$, $\sigma_s > \sigma_{s+1}$. Let $\mathbf{g}_{\mu}(j)$ be the sub-matrix of $\mathbf{g}$, which consists of the rows corresponding to all the basis elements μu_i such that $m_i = n_j$. We shall call $\mathbf{g}_{\mu}(j)$ a *row-strip of type* μ. Let $\mathbf{g}_{\Delta}(t)$ be the sub-matrix of $\mathbf{g}$ which corresponds to all the basis elements $\Delta^* \rho_s v_s$ such that $\rho_s = \sigma_t$. We call $\mathbf{g}_{\Delta}(t)$ a *row-strip of type* Δ. We order these row-strips so that $\mathbf{g}_{\Delta}(t)$ precedes $\mathbf{g}_{\Delta}(t+1)$ and $\mathbf{g}_{\mu}(j)$ precedes $\mathbf{g}_{\mu}(j+1)$ and also each $\mathbf{g}_{\Delta}(t)$. Let $\mathbf{r}^1, \mathbf{r}^2,...$ be the row-strips, of both types, arranged in this order. We define *column-strips* $\mathbf{c}_1, \mathbf{c}_2,...$, of types μ and Δ, in the same way, except that the inequalities analogous to $n_j < n_{j+1}$, $\sigma_s > \sigma_{s+1}$ are reversed and the column-strips of type Δ precede those of type μ.

Let T^{α}_{β} be the rank of the sub-matrix in which the union of the row-strips $\mathbf{r}^1,..., \mathbf{r}^{\alpha}$ intersects the union of the column-strips $\mathbf{c}_1,..., \mathbf{c}_{\beta}$. Then

$$\tau^{\alpha}_{\beta} = T^{\alpha}_{\beta} - T^{\alpha-1}_{\beta} - T^{\alpha}_{\beta-1} + T^{\alpha-1}_{\beta-1} \quad (T^0_j = T^i_0 = 0).$$

The invariance of T^{α}_{β}, and hence of τ^{α}_{β}, follows from (2.2) and the analogous set of conditions on automorphisms of $H^{n+k}(2)$. For it follows from these conditions that the transformation $\mathbf{g} \to \mathbf{f}^{-1}\mathbf{g}\mathbf{f}'$ is the resultant of elementary transformations in which a row in $\mathbf{r}^i$ (column in $\mathbf{c}_i$) may be added (mod 2) to a row in $\mathbf{r}^j$ (column in $\mathbf{c}_j$) if and only if $j \geqslant i$. As in (**1**), the matrix $\mathbf{g}$ may be reduced by these elementary transformations to a normal form in which each row or column contains at most one unity. When $\mathbf{g}$ is of this form, τ^{α}_{β} is the rank of the block $\mathbf{r}^{\alpha}.\mathbf{c}_{\beta}$ in which $\mathbf{r}^{\alpha}$ intersects $\mathbf{c}_{\beta}$. Moreover the normalization is completed by arranging that the units, if any, in each block $\mathbf{r}^{\alpha}.\mathbf{c}_{\beta}$ occur in the first τ^{α}_{β} rows and columns of $\mathbf{r}^{\alpha}.\mathbf{c}_{\beta}$, which do not contain units in any block $\mathbf{r}^{\lambda}.\mathbf{c}_{\mu}$ for $\lambda < \alpha$, $\mu < \beta$. It follows from arguments similar to those at the beginning of § 1 in (**1**) that the block invariants are invariant under (μ, Δ, γ)-isomorphisms. Therefore, if H is the system determined by a (finite) complex K, they are homotopy invariants of K.

† $\mathbf{f}^{-1}\mathbf{g}\mathbf{f}'$ is the matrix of $\phi'\gamma\phi^{-1}$.

In general, we cannot simultaneously reduce all the matrices $\mathbf{g}_k^n$ to normal form. For the reduction of $\mathbf{g}_j^m$, $\mathbf{g}_k^n$ may require different automorphisms of H^n, if m or $m+j$ is the same as $n-1$ or n, or of H^{n+k}, if m or $m+j$ is $n+k-1$ or $n+k$. But a sub-set $\mathbf{g}_j^m$, $\mathbf{g}_k^n$,... of these matrices can be simultaneously reduced to normal form if the pairs (m,j), (n,k),... are suitably spaced. Let this be so and let H, H' be two (μ, Δ, γ)-systems such that H^r, H'^r have the same rank and prime-power invariants ($r = 0, 1,...$) and H, H' have the same block invariants $\tau_{j\sigma}^{m\rho}$, $\tau_{k\beta}^{n\alpha}$,... . Then there are (μ, Δ)-bases for $H^r(2)$, $H'^r(2)$ ($r = 0, 1,...$), referred to which each of γ_j^m, γ_k^n,... is represented by the same (normalized) mod 2 matrix in both systems. Therefore there is a (μ, Δ)-isomorphism, $H \approx H'$, which commutes with each of γ_j^m, γ_k^n,... .

3. Two combinatorial operations

Let K be a finite cell-complex and let K^n be its n-section. Let $n > 0$ and let K_n be the complex formed by shrinking K^{n-1} to a point e^0 which is not in $K-K^{n-1}$ and is the single 0-cell of K_n. The two combinatorial operations consist of replacing K by K^n and by K_n. They are, in some respects, dual to each other. For example, if $1 \leqslant p < n < q$ and if $\dim K < 2q-1$, then

$$i_\sharp\colon \pi_p(K^n) \approx \pi_p(K), \qquad \iota^\sharp\colon \pi^q(K_n) \approx \pi^q(K),$$

where π_p, π^q indicate homotopy and cohomotopy† groups and $i_\sharp$, $\iota^\sharp$ are induced by the identical map $i\colon K^n \to K$ and the identification map $\iota\colon K \to K_n$. The purpose of this section is to show how the block invariants of K^n, K_n may be obtained from those of K.

We first consider K_n. Let H be the (μ, Δ, γ)-system of K and let G, consisting of groups G^r, $G^r(2)$, be the (μ, Δ, γ)-system of K_n. Let $\iota^*\colon G \to H$ be the homomorphism induced by $\iota\colon K \to K_n$. Since $\iota | K-K^{n-1}$ is a homeomorphism onto $K_n - e^0$ which maps each cell of $K-K^{n-1}$ on a cell of $K_n - e^0$, it follows that $\iota^* G^n = H^n$, $\iota^*\colon G^r \approx H^r$ if $r > n$. Moreover G^n is a free Abelian group since K_n has no $(n-1)$-dimensional torsion. Let $j_r^* = \iota^* | G^r(2)$ and let $\Delta_g^*\colon {}_2G^{n+1} \to G^n(2)$ be a right inverse of $\Delta\colon G^n(2) \to G^{n+1}$. Let $\Delta_h^*\colon {}_2H^{n+1} \to H^n(2)$ be defined by $\Delta_h^* \iota^* u = j_n^* \Delta_g^* u$ ($u \in {}_2G^{n+1}$). Then Δ_h^* is a right inverse of $\Delta\colon H^n(2) \to H^{n+1}$ and j_n^* maps the summand $\Delta_g^*({}_2G^{n+1}) \subset G^n(2)$ isomorphically on to $\Delta_h^*({}_2H^{n+1})$. Also $j_n^* \mu G^n \subset \mu H^n$ since $j_n^* \mu = \mu \iota^*$. Since μG^n and μH^n are (free) mod 2

† See (4).

modules and since $\iota^* G^n = H^n$, it follows that μG^n is a direct sum $A+B$, where $A = j_n^{*-1}(0)$ and $j_n^*|B$ is an isomorphism onto μH^n. Therefore

$$G^n(2) = A+B+C,$$

where $C = \Delta_g^*({}_2G^{n+1})$ and

$$j_n^*|(B+C)\colon B+C \approx H^n(2).$$

Let $\sigma_\beta^\alpha = \sigma_{k\beta}^{p\alpha}$ be the block invariants of K_n, for given values of p, k, such that $0 < k \leqslant p$ and $k \equiv 0 \pmod 2$. Let $\tau_\beta^\alpha = \tau_{k\beta}^{p\alpha}$ be the corresponding invariants of K. Obviously $\sigma_\beta^\alpha = 0$ if $p < n$, and $\sigma_\beta^\alpha = \tau_\beta^\alpha$ if $p > n$. Let $p = n$, let $m = n+k$, and let

$$\gamma' = \gamma_k^n\colon G^n(2) \to G^m(2), \qquad \gamma = \gamma_k^n\colon H^n(2) \to H^m(2).$$

Then $j_m^* \gamma' A = \gamma j_n^* A = 0$ and, obviously, $j_m^{*-1}(0) = 0$. Therefore $\gamma' A = 0$. Let x_i, y_s mean the same as in (2.1) and let $\{a_h\}$ be a basis for A. Then $\{a_h, b_i, c_s\}$ is a (μ, Δ)-basis for $G^n(2)$, where $b_i \in B$, $c_s \in C$ are such that

$$x_i = j_n^* b_i, \qquad y_s = j_n^* c_s.$$

Let $\{d_u\}$ be a (μ, Δ)-basis for $G^m(2)$. Then $\{j_m^* d_u\}$ is obviously a (μ, Δ)-basis for $H^m(2)$.

Let $\mathfrak{g}$, $\mathfrak{g}'$ be the matrices which, with these bases, define γ, γ'. Let $\mathbf{r}^1,\dots, \mathbf{r}^l$ be the row-strips of type† μ in $\mathfrak{g}$. Since $\gamma' = j_m^{*-1}\gamma j_n^*$ and $\gamma' A = 0$, the matrix $\mathfrak{g}'$ consists of $\mathfrak{g}$, augmented by an initial block of rows of zeros if $A \neq 0$. Since G^n is free Abelian, $\mathfrak{g}'$ has only one row-strip of type μ. Therefore

$$\sigma_\beta^1 = \mathrm{T}_\beta^l - \mathrm{T}_{\beta-1}^l = \sum_{\lambda=1}^{l} \tau_\beta^\lambda, \qquad \sigma_\beta^{1+\alpha} = \tau_\beta^{l+\alpha} \quad (\alpha \geqslant 1). \tag{3.1}$$

Thus the block invariants of K_n are uniquely determined by those of K, subject to the convention that σ_β^1 is defined as zero if $G^n = 0$. Evidently the only numerical invariant of K_n, among those we are considering, which is not determined by the invariants of K is the nth Betti number $p_n(K_n)$, which is the rank of G^n.

We now consider K^m. Let F, consisting of groups F^r, $F^r(2)$, be the (μ, Δ, γ)-system of K^m. Obviously $i_r^*\colon H^r \approx F^r$ if $r < m$, where i_r^* is induced by the identical map $i\colon K^m \to K$. On considering the normal

† We make the convention that, if $\mu H^n = 0$, then $\mathfrak{g}$ has a vacuous row-strip of type μ and $\tau_\beta^1 = 0$.

form of the incidence matrix for $\delta\colon C^m \to C^{m+1}$, where C^r is the group of integral r-co-chains in K, we see that

(i) $i^*_m\colon H^m \to F^m$ is an isomorphism into F^m, such that $i^*_m H^m_0 = F^m_0$, where H^m_0, F^m_0 are the maximal finite sub-groups of H^m, F^m;

(ii) $j^*_m \Delta^*({}_2H^{m+1}) = \mu F^m_1$, where $j^*_m\colon H^m(2) \to F^m(2)$ is induced by i and F^m_1 is a free summand† of F^m.

Moreover $j^{*-1}_m(0) = 0$ and $F^m(2)$ is the direct sum

$$F^m(2) = j^*_m H^m(2) + \mu F^m_2,$$

where F^m_2 is another free summand of F^m.

Let $\rho^\alpha_\beta = \rho^{n\alpha}_{k\beta}$ be the block invariants of K^m and let $\tau^\alpha_\beta = \tau^{n\alpha}_{k\beta}$. Obviously $\rho^\alpha_\beta = 0$ if $n+k > m$, and $\rho^\alpha_\beta = \tau^\alpha_\beta$ if $n+k < m$. Let $n+k = m$, let $\mathbf{c}_1,\ldots, \mathbf{c}_{r-1}$ be the column-strips of type Δ in $\mathfrak{g}$, and let $\mathbf{c}_r$ be the column-strip of type μ which corresponds to the free generators‡ of H^m. Since $F^{m+1} = 0$, the first column strip in the matrix of $\gamma\colon F^n(2) \to F^m(2)$ is the one of type μ which corresponds to the free generators of F^m. Therefore it follows from arguments which are similar to those leading to (3.1) that

$$\rho^\alpha_1 = \sum_{\lambda=1}^{r} \tau^\alpha_\lambda, \qquad \rho^\alpha_{1+\beta} = \tau^\alpha_{r+\beta} \quad (\beta \geqslant 1). \tag{3.2}$$

Thus the block invariants of K^m are uniquely determined by those of K, subject to the convention that $\rho^\alpha_1 = 0$ if the group F^m is finite. Clearly the only numerical invariant of K^m, among those we are considering, which is not determined by the invariants of K is $p_m(K^m)$, which is the rank of F^m.

Let $n > 2$, $k = 2$ and let $\dim K \leqslant n+2$. Then the structure of $\pi^{n+1}(K)$ is determined by the numerical invariants of K_n, as shown in § 5 of (**1**). Let N be a normal A^2_n-complex of the same homotopy type as K_n. Then N consists of certain elementary A^2_n-complexes, which are attached to each other at the single 0-cell of N. The structure of N is determined by the numerical invariants of K_n, according to (3.2) and (3.3) in (**1**). Let K'_n have the same numerical invariants as K_n, except for p_n, which is Δ_0 in (**1**), and let N' be a normal A^2_n-complex of the same homotopy type as K'_n. Let $p_n(K'_n) = p_n(K_n)+q$ $(q > 0)$. Then it follows from (3.3) in (**1**) that

$$N' = N \cup S^n_1 \cup \ldots \cup S^n_q.$$

† F^m_1 arises from the basic co-chains $c \in C^m$ such that $\delta c = 2\rho c'$ $(\rho > 0)$ and F^m_2, below, from those with $\delta c = (2l+1)c'$ $(l \geqslant 0)$.

‡ $\mathbf{c}_r$ may be vacuous, in which case $\tau^\alpha_r = 0$.

Therefore $\pi^{n+1}(N') \approx \pi^{n+1}(N)$ and it follows that the structure of $\pi^{n+1}(K)$ is determined by the numerical invariants of K.

Similarly the structure of $\pi_{n+1}(K)$ is determined by the numerical invariants of K if $\pi_r(K) = 0$ for $r = 1,..., n-1$ $(n > 2)$, $\dim K$ being arbitrary.

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THE ASYMPTOTIC BEHAVIOUR OF A KIND OF MULTIPLE INTEGRALS INVOLVING A PARAMETER

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1. Introduction

THE object of this paper is to investigate the asymptotic behaviour, as $\lambda \to \infty$ through positive values, of an n-fold integral of the form

$$I(\lambda) = \int \underset{R}{\cdots} \int F(x_1,..., x_n; \lambda)\, dx_1...dx_n.$$

Here F is a positive real-valued function of $x = (x_1,..., x_n)$ defined in R, and R is a simply connected, finitely bounded, n-dimensional domain in Euclidean n-space.

My result is a generalization of Laplace's classical theorem on the asymptotic expression of an integral [see (**1**) 77–82, (**2**) 277–80, (**3**), (**4**), (**5**) 181–2]. I suppose that, for any $\lambda > 0$, the integrand

$$F(x_1,..., x_n; \lambda) = F(x; \lambda) = \exp[f(x; \lambda)]$$

attains an absolute maximum at an interior point

$$x = (x_1,..., x_n) = (\phi_1(\lambda),..., \phi_n(\lambda))$$

of R. The case in which $f(x; \lambda) = \lambda g(x)$ and the function

$$g(x) = g(x_1,..., x_n)$$

takes a maximum value at a boundary point of the closed domain R has been discussed in a previous paper (**9**). But the method of (**9**) cannot be applied unaltered to the present case, because $(\phi_1(\lambda),..., \phi_n(\lambda))$ is now a variable point depending on the parameter λ, and moreover the limit $(\lim \phi_1(\lambda),..., \lim \phi_n(\lambda))$ may or may not exist. Thus in the present investigation a modification of the methods of Laplace and Haviland (**6**) will be employed.

2. Notation and terminology

Throughout the paper f_k denotes the partial derivative of $f(x_1,..., x_n; \lambda)$ with respect to x_k. Similarly, f_{ik} denotes the second-order partial derivative of f with respect to x_k and x_i. If $f(x_1,..., x_n; \lambda)$ and all its partial derivatives $\partial f/\partial \lambda$, f_k, and f_{ik} are continuous functions with respect to both $(x_1,..., x_n)$ and λ with $x \in R$ $(0 < \lambda < \infty)$, then $f(x_1,..., x_n; \lambda)$ is called a *function of class* C^2 with *parameter* λ.

By a single letter such as x or ξ, I always mean a point $(x_1,..., x_n)$ or $(\xi_1,..., \xi_n)$ in the Euclidean n-space. Moreover, $H_k[-f]$ denotes the Hessian of $-f(x_1,..., x_n; \lambda)$ with regard to the first k variables $x_1,..., x_k$; namely,

$$H_k[-f] = \begin{vmatrix} -f_{11}(x;\lambda) & . & . & . & -f_{1k}(x;\lambda) \\ . & . & . & . & . \\ -f_{k1}(x;\lambda) & . & . & . & -f_{kk}(x;\lambda) \end{vmatrix} \quad (k = 1,..., n).$$

By $(x;\lambda) \to (\xi;\infty)$ I mean that x, λ tend to ξ, ∞ respectively and independently. By $f(x;\lambda) = O(g(x;\lambda))$ I mean that there is a positive number N such that $|f/g| < N$ as† $(x;\lambda) \to (\xi;\infty)$. Similarly $f = o(g)$ means that $|f/g| \to 0$. If there exist $N_1 > 0$, $N_2 > 0$ such that

$$N_1 < |f/g| < N_2$$

as $(x;\lambda) \to (\xi;\infty)$, I write $f \asymp g$. Similarly we may write $f \sim g$, if $f/g \to 1$.

As a convention I assume that any statement involving index i or k is true for all i, $k = 1,..., n$. Thus, for example, $f_k = 0$ represents not a single equation but a system of simultaneous equations $f_k = 0$ $(k = 1,..., n)$.

3. Statement of theorems

The principal result is contained in the following

THEOREM 1. *Let $f(x;\lambda)$ be a real function of class C^2 with parameter λ and $x \in R$ such that*

(i) *$f_k(x;\lambda) = 0$ have at least one solution for $\lambda > N$, and there is an interior point $\xi \in R$ satisfying $\lim_{\lambda\to\infty} f_k(\xi;\lambda) = 0$;*

(ii) *$H_k[-f] > 0$ hold throughout R for $\lambda > N$;*

(iii) *$f_{ik}(x;\lambda) \sim f_{ik}(\xi;\lambda)$ and $-f_{ik}(x;\lambda) \to \infty$ as $(x;\lambda) \to (\xi;\infty)$;*

(iv) *$f_{ik}(x;\lambda) \asymp f_{11}(x;\lambda)$ and $H_k[-f] \asymp (-f_{11}(x;\lambda))^k$ as $(x;\lambda) \to (\xi;\infty)$, Then for λ large we have*

$$\int_R \cdots \int \exp[f(x;\lambda)]\, dx_1...dx_n \sim \exp[f(\xi;\lambda)] \left(\frac{(2\pi)^n}{H_n[-f(\xi;\lambda)]}\right)^{\frac{1}{2}}. \qquad (1)$$

It will be observed from the proof that all the conditions imposed on $f(x;\lambda)$ are required only to be valid in a small neighbourhood of $x = \xi$. Thus the domain of integration R may be replaced by a larger region D. provided that

(v) $\overline{\text{bound}}\, f(y;\lambda) \leqslant \underline{\text{bound}}\, f(x;\lambda)$ for $x \in R$, $y = (y_1,..., y_n) \in D-R$ and $\lambda > 0$. On the contrary, if D is given, the neighbourhood R of ξ

† In other words, there exists a number $\delta > 0$ and a large number $M > 0$ such that $|f/g| < N$, whenever $|x_k - \xi_k| < \delta$, $\lambda > M$. If for any given $\epsilon > 0$, $|f/g| < \epsilon$ whenever $|x_k - \xi_k| < \delta(\epsilon)$, $\lambda > M(\epsilon)$, then I write $f = o(g)$.

may be taken as small as we like. In fact the right-hand side of (1) does not involve R. Thus it is easy to obtain the following

THEOREM 2. *If the functions* $\phi(x)$, $\phi(x)\exp[f(x;\lambda)]$ *are absolutely integrable over* D *for every* $\lambda > 0$, *if* $f(x;\lambda)$ *satisfies all the hypotheses of Theorem* 1 *together with* (v) *in which* R *is a neighbourhood of* ξ, *and if* $\phi(x)$ *is continuous at* $x = \xi$ *with* $\phi(\xi) \neq 0$, *then we have*

$$\int_D \cdots \int \phi(x)\exp[f(x;\lambda)]\,dx_1 \dots dx_n \sim \phi(\xi)\exp[f(\xi;\lambda)]\left(\frac{(2\pi)^n}{H_n[-f(\xi;\lambda)]}\right)^{\frac{1}{2}}. \quad (2)$$

It is clear that the 1-dimensional case of Theorem 2 includes the Laplace asymptotic theorem as a special case. For, taking $n = 1$, $f(x;\lambda) = \lambda h(x)$, the classical asymptotic formula follows from (2). Note that condition (iii) of Theorem 1 just corresponds to the requirement

$$\lambda h''(x) \sim \lambda h''(\xi), \qquad f_{11}(x;\lambda) = \lambda h''(x) \to -\infty \quad \text{as} \quad (x;\lambda) \to (\xi;\infty)$$

in the classical case. The most stringent condition is (iv), which may be replaced by some weaker hypothesis. In fact we have the following more general result.

THEOREM 3. *If there is a continuous function* $h(x;\lambda)$ *such that*

$$f_{ik}(x;\lambda) = O(h(x;\lambda))$$

and

$$\text{(iv}\,a) \quad \log H_n[-f] = o\left(\frac{H_k[-f]}{H_{k-1}[-f]}\right), \qquad \log H_n[-f] = o\left(\frac{H_n[-f]}{(h(x;\lambda))^{n-1}}\right),$$

as $(x;\lambda) \to (\xi;\infty)$, *where* $H_0[-f] = 1$, $H_n[-f] \to \infty$, *then the asymptotic formula* (1) *of Theorem* 1 *is true under the conditions* (i), (ii), (iii), (iv a).

Obviously Theorem 1 is implied by Theorem 3. For condition (iv) is a special case of (iv a) with $h = f_{11}$.

4. Preliminary lemmas

For proof of the theorems, we need some preliminary lemmas.

LEMMA 1. *Let* $A = [a_{ik}]$ *be a square symmetric matrix of a positive definite form in* n *variables and let* $x = (x_1,\dots, x_n)$ *denote a row-vector. Then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}xAx')\,dx_1 \dots dx_n = (2\pi)^{\frac{1}{2}n}|A|^{-\frac{1}{2}}, \quad (3)$$

where x' *denotes the transpose of* x *and* $|A|$ *is the determinant of* A.

This lemma is well known [see, for example, (7) 175].

LEMMA 2. *Let* $A = [a_{ik}]$ *be a square symmetric matrix of the definite form* xAx'. *Then*

$$xAx' = \sum_{k=1}^{n} (A_{k-1}^{(k-1)}A_k^{(k)})^{-1}(A_k^{(k)}x_k + A_{k+1}^{(k)}x_{k+1} + \dots + A_n^{(k)}x_n)^2, \quad (4)$$

where $A_0^{(0)} = 1$, $A_s^{(1)} = a_{s1}$ *and*

$$A_s^{(k)} = \begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{s-1\,1} & \cdot & \cdot & \cdot & a_{s-1\,k} \\ a_{s1} & \cdot & \cdot & \cdot & a_{sk} \end{vmatrix} \qquad (1 \leqslant k \leqslant s \leqslant n).$$

The explicit expression (4) is a direct consequence of Darboux's reduction of quadratic forms [(**8**) 191–2]. It may also be deduced by repeated application of Lagrange's reduction.

In what follows I use certain geometrical terms for convenience. 'Distance', 'neighbourhood', 'diameter', 'convexity', 'directional derivatives', etc., have their usual meaning in Euclidean n-space. Thus, for example, a neighbourhood $U(\xi)$ of ξ with diameter δ is a point set of the n-space in which ξ is an interior point of U and the greatest distance between two points of the closure $\bar{U}$ is the number $\delta > 0$. Similarly, a hypersurface defined by $u = f(x_1,..., x_n)$ is said to be *concave downward* in R, if the second-order directional derivative of u is negative definite throughout R.

LEMMA 3. *Let conditions* (i) *and* (ii) *of Theorem* 1 *be fulfilled. Then there exists a region* R' *and* n *single-valued continuous functions* $x_k = \phi_k(\lambda)$ *with* $(\phi_1(\lambda),..., \phi_n(\lambda)) \in R' \supseteq R$ *such that, for* $\lambda > N$,

$$f_k(\phi_1(\lambda),..., \phi_n(\lambda); \lambda) \equiv 0, \tag{5}$$

and that, for every fixed $\lambda > N$, *the hypersurface defined by* $u = f(x_1,..., x_n; \lambda)$ *is concave downward in* R.

Condition (i) implies the existence of a point $x = \bar{x} \in R$ such that $f_k(\bar{x}; \lambda) = 0$ for a certain fixed $\lambda > N$. By condition (ii) and the fact that $f(x; \lambda)$ belongs to C^2 we see that the implicit-function theorem is applicable to the system of equations $f_k = 0$. Thus there exists an interval I: $M_1 < \lambda < M_2$ $(M_1 > N)$ and a neighbourhood U of $\bar{x}$ such that there are n single-valued continuous functions $\phi_k(\lambda)$ defined in I and satisfying (5) with $(\phi_1(\lambda),..., \phi_n(\lambda)) \in U$. Note that $H_k[-f] > 0$ throughout R and for every $\lambda > N$. Thus by continued application of the implicit-function theorem or by the principle of analytic continuation for real functions we see that the definitions of $\phi_k(\lambda)$ can be extended to all $\lambda > N$, where the region of existence of $(\phi_1(\lambda),..., \phi_n(\lambda))$ may sometimes cover R. Hence the first part of Lemma 3 follows.

To prove the second part of the lemma, take an arbitrary line of the Euclidean n-space, say $\Gamma_{(\theta_1,...,\theta_n)}$, with direction cosines $\cos\theta_1,..., \cos\theta_n$,

where θ_k is the angle made by Γ and the x_k-axis, so that $\sum\cos^2\theta_k = 1$ and the equation of Γ may be written as

$$(x_1-x_1^0)/\cos\theta_1 = (x_2-x_2^0)/\cos\theta_2 = \ldots = (x_n-x_n^0)/\cos\theta_n.$$

It is easily found that the first and second directional derivatives of $u = f(x_1,\ldots,x_n;\lambda)$ with respect to Γ_θ are as follows:

$$\frac{\partial u}{\partial\Gamma_\theta} = \sum_{k=1}^{n} f_k\cos\theta_k, \qquad \frac{\partial^2 u}{\partial\Gamma_\theta^2} = zHz', \tag{6}$$

where $H = [f_{ik}]$ and

$$z = (z_1,\ldots,z_n) = (\cos\theta_1,\ldots,\cos\theta_n).$$

Hence, applying Lemma 2 and using the condition (ii), we find that

$$zHz' < 0, \tag{7}$$

whenever $\lambda > N$. The lemma is therefore proved.

LEMMA 4. *Let the conditions of Theorem* 1 *be fulfilled. Then there are* n *single-valued continuous functions* $x_1 = \phi_1(\lambda),\ldots, x_n = \phi_n(\lambda)$ *satisfying the equations of* (5) *such that*

$$\lim_{\lambda\to\infty}\phi_k(\lambda) = \xi_k. \tag{8}$$

Let ϵ be an arbitrary positive number. Then by condition (i), we have

$$|f_k(\xi;\lambda)| < \epsilon, \tag{9}$$

for $\lambda > N'(\epsilon)$, where $N'(\epsilon)$ is a positive number depending on ϵ. It is no restriction to assume $N' > N$. From (5) and by the continuity of $f_1,\ldots,f_n$ we have

$$|f_k(x;\lambda)| < \epsilon \tag{10}$$

whenever $x \in U(\phi_1(\lambda),\ldots,\phi_n(\lambda))$, where U is a maximal neighbourhood of $x = (\phi_1(\lambda),\ldots,\phi_n(\lambda))$ with diameter $\delta = \delta(\lambda)$. From Lemma 3 we have seen that, for each fixed $\lambda > N$, $u = f(x_1,\ldots,x_n;\lambda)$ is a hypersurface having a maximum at $(\phi_1(\lambda),\ldots,\phi_n(\lambda))$, and that the surface is concave downward in R. Hence (10) is valid only in the maximal neighbourhood indicated. A comparison of (9) and (10) gives

$$\xi = (\xi_1,\ldots,\xi_n) \in U\{(\phi_1(\lambda),\ldots,\phi_n(\lambda)\} \quad (\lambda > N'(\epsilon)). \tag{11}$$

Let us now define $\rho(\lambda)$ to be the maximal radius measured from ξ to the most distant point of the boundary of $U\{(\phi_1(\lambda),\ldots,\phi_n(\lambda))\} = U\{\phi(\lambda)\}$ so that

$$\rho(\lambda) \leqslant \delta(\lambda) \leqslant 2\rho(\lambda).$$

What I shall prove in the following is that $\rho(\lambda)\to 0$ as $\lambda\to\infty$. Now suppose on the contrary that $\rho(\lambda)$ does not tend to zero. Then there exists at least a number $d > 0$ and a sequence $\{\lambda_\nu\}$ such that

$$\rho(\lambda_\nu) \geqslant d, \qquad \lambda_1 < \lambda_2 < \ldots < \lambda_\nu < \ldots, \qquad \lambda_\nu\to\infty \quad (\lambda_1 > N').$$

Thus corresponding to $\{\lambda_\nu\}$ we have a sequence of maximal neighbourhoods of ξ, $\{U_\nu(\phi(\lambda_\nu))\}$ say, and a sequence of radii, $\{\rho(\lambda_\nu)\}$ say.

The line segment determined by ξ and the most distant boundary point of U_ν from ξ may be called a *line segment* of $\rho(\lambda_\nu)$. I call

$$\theta^{(\nu)} = (\theta_1^{(\nu)},..., \theta_n^{(\nu)})$$

the *directional angle* of $\rho(\lambda_\nu)$ if the line segment of $\rho(\lambda_\nu)$ has direction cosines $\cos\theta_1^{(\nu)},..., \cos\theta_n^{(\nu)}$, where $\theta_n^{(\nu)}$ is chosen to be the smallest possible angle. Thus, corresponding to $\{\rho(\lambda_\nu)\}$ we have a sequence $\{\theta^{(\nu)}\}$.

Applying Lemma 2 to the second equation of (6), we have

$$-\frac{\partial^2 u}{\partial\Gamma_\theta^2} = \sum_{k=1}^{n} (H_{k-1}[-f]H_k[-f])^{-1}(H_k^{(k)}\cos\theta_k + ... + H_n^{(k)}\cos\theta_n)^2 > 0, \tag{12}$$

where $H_s^{(k)} = A_s^{(k)}$ in which $a_{ik} = f_{ik}$ (see Lemma 2). It is known that the axes (or basis) of the $(x_1,..., x_n)$-coordinate system can be rotated by means of a unimodular orthogonal linear transformation (whose determinant is 1), so that it is always possible to replace θ_n by 0. For an arbitrary $\lambda > N'$, let θ be the directional angle of $\rho(\lambda)$ and let T be the required orthogonal transformation (i.e. $y = xT$, $TT' = I$) under which

$$(x_1,..., x_n) \xrightarrow{T} (y_1,..., y_n), \qquad u = f(x;\lambda) \xrightarrow{T} g(y;\lambda) = v,$$

$$(\xi_1,..., \xi_n) \xrightarrow{T} (\eta_1,..., \eta_n), \qquad \Gamma_{(\theta_1,...,\theta_n)} \xrightarrow{T} \Gamma_{(\phi_1,...,\phi_n)},$$

where $\phi_n = 0$. Note that the directional derivatives of u as well as the hypersurface remain unchanged under the transformation (rotation) T, so that by Lemma 2 we have

$$-\frac{\partial^2 u}{\partial\Gamma_\theta^2} = -\frac{\partial^2 v}{\partial\Gamma_\phi^2}$$

$$= \sum_{k=1}^{n} (H_{k-1}[-g]H_k[-g])^{-1}(G_k^{(k)}\cos\phi_k + ... + G_n^{(k)}\cos\phi_n)^2 > 0, \tag{13}$$

where $\cos\phi_n = 1$ and $G_s^{(k)} = A_s^{(k)}$ in which $a_{ik} = g_{ik} = g_{ik}(y;\lambda)$.

Since the Hessian is an invariant form under the group of unimodular linear transformations [see, for example, (**6**) 627], we have

$$H_n[-g] = H_n[-f].$$

Hence the definite quadratic form of (13) contains the positive term

$$(H_{n-1}[-g]H_n[-g])^{-1}(H_n[-g])^2 = H_n[-f]/H_{n-1}[-g]. \tag{14}$$

By differentiation, each element g_{ik} of $H_{n-1}[-g]$ can be expressed linearly in terms of $f_{ik}(x;\lambda)$'s. It is always possible to assume that T^{-1}

(as well as T) is an orthogonal transformation (matrix) having row-vectors of unit length, so that

$$yT^{-1} = x, \qquad T^{-1}(T^{-1})' = I,$$

$$\pm g_{ik}(y;\lambda) \leqslant \sum_{k=1}^{n} |f_{kk}(x;\lambda)|+2\sum_{i<k} |f_{ik}(x;\lambda)|. \tag{15}$$

Hence by conditions (iii), (iv), (15) and the continuity of $H_n[-f]$, we can determine a *fixed neighbourhood*† $V(\xi)$ and a number $N'' > N'$ such that

$$\pm g_{ik}(y;\lambda) < K|f_{11}(\xi;\lambda)|,$$

$$H_n[-f(x;\lambda)] > \tfrac{1}{2}H_n[-f(\xi;\lambda)], \tag{16}$$

for $\lambda > N''$, $x \in V(\xi)$, where K is a positive number independent of T.

Returning now to the sequence $\{\lambda_\nu\}$, let T_ν be the required transformation corresponding to $\rho(\lambda_\nu)$ or $\theta^{(\nu)}$ ($\nu = 1, 2, 3,...$), so that

$$f(x;\lambda_\nu) = g^{(\nu)}(y;\lambda_\nu)$$

with $\cos\phi_n^{(\nu)} = 1$ under T_ν. Since the validity of (16) is independent of T_ν, it follows from (iii), (iv) that, for any fixed $x \in V(\xi)$ we have

$$\frac{H_n[-f(x;\lambda_\nu)]}{H_{n-1}[-g^{(\nu)}(y;\lambda_\nu)]} > \frac{H_n[-f(\xi;\lambda_\nu)]}{2(n-1)!\{K|f_{11}(\xi;\lambda_\nu)|\}^{n-1}} \to \infty \tag{17}$$

as $\lambda_\nu \to \infty$.

Let W_ν be a neighbourhood of ξ defined by the intersection:

$$W_\nu(\xi) = U_\nu\{\phi(\lambda_\nu)\} \cap V(\xi) \quad (\nu = 1, 2, 3,...). \tag{18}$$

Then from (13), (14), (17) we may infer that, for any given $M > 0$, there is a large number N''' ($> N''$) depending on M such that

$$-\left(\frac{\partial^2 u}{\partial\Gamma^2_{\theta^{(\nu)}}}\right) > M \tag{19}$$

whenever $\lambda_\nu > N'''$, $x \in W_\nu(\xi)$. Let d' be the shortest distance from ξ to the boundary of $V(\xi)$, and let

$$\min(\tfrac{1}{2}d, \tfrac{1}{2}d') = d^*. \tag{20}$$

Evidently d^* is a positive number independent of M, ν, etc. Since $U_\nu\{\phi(\lambda_\nu)\}$ has a maximal radius $\rho(\lambda_\nu) \geqslant d$, it is easy to see that for every $\lambda_\nu > N'''$, the neighbourhood $W_\nu(\xi)$ always contains an interior point‡ which has a distance d^* from ξ.

† It is easy to see that the existence of such a fixed neighbourhood is a very important step in the present proof of Lemma 4.

‡ Note that there are only two possible cases, either $U_\nu\{\phi(\lambda_\nu)\}$ and $V(\xi)$ intersect properly or one is completely contained in the other. The latter case is trivial in view of (20). The first case is also evident if we notice that $W_\nu(\xi)$ has at least a boundary point which is a boundary point of $V(\xi)$.

Let us now take M so large that $Md^* > 2n\epsilon$ when $\lambda_\nu > N''' = N'''(M)$, $x \in W_\nu(\xi)$. Moreover, for a fixed ν, let ξ^* be a point of $W_\nu(\xi)$ having a distance d^* from ξ. Then from (19), it follows that

$$\left|\int \frac{\partial^2 u}{\partial \Gamma^2_{\theta^{(\nu)}}}\, ds\right| > Md^* > 2n\epsilon, \tag{21}$$

where the integral is taken from ξ to ξ^* along $\Gamma_{\theta^{(\nu)}}$, ν being a fixed index with $\lambda_\nu > N'''$. But, on the other hand, by (6) and (10), we have

$$\begin{aligned}\left|\left(\frac{\partial u}{\partial \Gamma_{\theta^{(\nu)}}}\right)_{x=\xi^*} - \left(\frac{\partial u}{\partial \Gamma_{\theta^{(\nu)}}}\right)_{x=\xi}\right| &= \left|\sum_{k=1}^{n} f_k(\xi^*;\lambda_\nu)\cos\theta_k^{(\nu)} - \sum_{k=1}^{n} f_k(\xi;\lambda_\nu)\cos\theta_k^{(\nu)}\right| \\ &\leqslant \sum_{k=1}^{n} (|f_k(\xi^*;\lambda_\nu)|+|f_k(\xi;\lambda_\nu)|) < 2n\epsilon \end{aligned}\tag{22}$$

for $\lambda_\nu > N''' > N'$. Thus the comparison of (21) and (22) leads to a contradiction, so that we must have $\rho(\lambda) \to 0$, and consequently $\delta(\lambda) \to 0$, as $\lambda \to \infty$. The lemma is therefore established by (11).

5. Proof of Theorem 1

By Lemma 3 we know that, for any fixed $\lambda > N$, the integrand $\exp[f(x;\lambda)]$ always assumes an absolute maximum at a point

$$x = (\phi_1(\lambda),..., \phi_n(\lambda)) \in R'.$$

Lemma 4 asserts that

$$\lim_{\lambda\to\infty} (\phi_1(\lambda),..., \phi_n(\lambda)) = (\xi_1,..., \xi_n) \in R.$$

Since ξ is an interior point of R, we can find a large number $M > N$ such that the variable point $(\phi_1(\lambda),..., \phi_n(\lambda))$ is always contained in R whenever $\lambda > M$.

Let us now keep λ ($> M$) fixed and write

$$\begin{aligned}\int_R \cdots \int \exp[f(x;\lambda)]\, dR &= \int_U \cdots \int \exp[f(x;\lambda)]\, dx + \int_{R-U} \cdots \int \exp[f(x;\lambda)]\, dx \\ &= I_1(\lambda)+I_2(\lambda),\end{aligned}$$

where U is a neighbourhood of $(\phi_1(\lambda),..., \phi_n(\lambda))$ whose volume depends on λ, $R-U$ denotes the complementary region of U with respect to the whole domain R. For convenience, we take $U = U_\epsilon$ to be a hypercube having the variable length $2\epsilon(\lambda)$ on each side and containing

$$(\phi_1(\lambda),..., \phi_n(\lambda))$$

as its centre. By Taylor's formula with remainder we have

$$\begin{aligned}J(\lambda) &= I_1(\lambda)\exp[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)] \\ &= \int_U \cdots \int \exp\left[\tfrac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} f_{ik}(X_1,..., X_n;\lambda)\{x_i-\phi_i(\lambda)\}\{x_k-\phi_k(\lambda)\}\right] dx_1...dx_n,\end{aligned}\tag{23}$$

where $$|X_k-\phi_k(\lambda)| < |x_k-\phi_k(\lambda)| \leqslant \epsilon(\lambda). \tag{24}$$

Since $X_1,..., X_n$ all depend on $x = (x_1,..., x_n)$, we may write $X_k = X_k(x)$. For fixed x, let us write

$$x_k-\phi_k(\lambda) = u_k, \qquad -f_{ik}(X_1,..., X_n; \lambda) = a_{ik}, \qquad [a_{ik}] = A.$$

Then the integral $J(\lambda)$ may be written as

$$J(\lambda) = \int\limits_{-\epsilon(\lambda)}^{\epsilon(\lambda)} ... \int\limits_{-\epsilon(\lambda)}^{\epsilon(\lambda)} \exp(-\tfrac{1}{2}uAu')\, du_1...du_n, \tag{25}$$

where $u = (u_1,..., u_n)$ and u' is the transpose of u.

For asymptotic evaluation of $J(\lambda)$ we need the following

LEMMA 5. *Let* $b_{ik} = -f_{ik}(\phi_1(\lambda),..., \phi_n(\lambda); \lambda)$ *and let* $[b_{ik}] = B$. *Then*

$$\lim_{u_k\to 0} (uAu')/(uBu') = 1. \tag{26}$$

From condition (ii) it is clear that uBu' is a positive definite form so that $uBu' > 0$. On the other hand, we see that the functional elements of A are continuous at $x = (\phi_1(\lambda),..., \phi_n(\lambda))$, i.e.

$$a_{ik} = -f_{ik}(X_1(x),..., X_n(x); \lambda) \to -f_{ik}(\phi_1(\lambda),..., \phi_n(\lambda); \lambda) = b_{ik}$$

as $x \to (\phi_1(\lambda),..., \phi_n(\lambda))$. Hence it follows that uAu' is continuous and $uAu' \to uBu'$ as $x \to (\phi_1(\lambda),..., \phi_n(\lambda))$ or correspondingly $u \to (0,..., 0)$. The lemma thus follows.

Suppose that $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$. Then from (24) it follows that $u \to (0,..., 0)$, so that by Lemma 5 we may write

$$uAu' = uBu'\{1+\epsilon'(u)\},$$

where $\epsilon'(u) \to 0$ with $\epsilon(\lambda)$. It is known that uBu' can be reduced to the sum of n squares by means of a congruent linear transformation: namely

$$uBu' = vT'BTv' = vv', \qquad v = (v_1,..., v_n)$$

where $u' = Tv'$, $T'BT = I$, and the Jacobian of the transformation is given by $|\partial(u)/\partial(v)| = |T| = |B|^{-\frac{1}{2}}$. Under the transformation T the neighbourhood U of u is transformed to a neighbourhood V of v. Set

$$M(\epsilon) = \max_{|u_k|\leqslant\epsilon}\{\epsilon'(u)\}, \qquad m(\epsilon) = \min_{|u_k|\leqslant\epsilon}\{\epsilon'(u)\}.$$

Clearly we have

$$J(\lambda) \leqslant |B|^{-\frac{1}{2}} \int ... \int_V \exp[-\tfrac{1}{2}vv'(1+m(\epsilon))]\, dv_1...dv_n = \bar{J},$$

$$J(\lambda) \geqslant |B|^{-\frac{1}{2}} \int ... \int_V \exp[-\tfrac{1}{2}vv'(1+M(\epsilon))]\, dv_1...dv_n = \underline{J}, \tag{27}$$

where v_k^2 can be written explicitly by means of Lemma 2, viz.

$$v_k^2 = (B_{k-1}^{(k-1)}B_k^{(k)})^{-1}(B_k^{(k)}u_k+...+B_n^{(k)}u_n)^2. \tag{28}$$

What I shall show in the next step is that there is a positive-valued function $\epsilon(\lambda)$ satisfying the conditions

(I) $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$;

(II) the integration limits (lower and upper limits) for each v_k of $\bar{J}$ or $\underline{J}$ tend to infinity with λ;

(III) $I_2(\lambda)/I_1(\lambda) \to 0$ as $\lambda \to \infty$.

For the purpose just mentioned we may take

$$\epsilon(\lambda) = K_0\left(\frac{\log H_n[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)]}{-f_{11}(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)}\right)^{\frac{1}{2}}, \tag{29}$$

where K_0 is a certain positive constant. It is clear that the condition (I) is satisfied. For, by condition (iv) of Theorem 1, we have

$$H_n[-f] \asymp (-f_{11})^n,$$

so that $\log H_n[-f] = o(-f_{11})$.

For verification of (II), we need only substitute $\pm\epsilon(\lambda)$ for u_k on the right-hand side of (28), e.g., if

$$u_k = \pm\epsilon(\lambda), \qquad u_{k+1} = ... = u_n = 0,$$

then (28), (29) together with condition (iv) imply that

$$v_k^2 = (B_{k-1}^{(k-1)}B_k^{(k)})^{-1}(B_k^{(k)}\epsilon(\lambda))^2 = \epsilon(\lambda)^2 H_k[-f]/H_{k-1}[-f]$$
$$\asymp \epsilon(\lambda)^2 H_1[-f] \asymp \log H_n[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)] \to \infty. \tag{30}$$

This shows that $|v_k|$ tends to infinity with λ. Since the previous linear transformation T is singular and leaves the convexity of the domain U_ϵ invariant, it is clear that the corresponding domain V $(= V_{\epsilon(\lambda)})$ of v will have no finite boundary or definite bounds as $\lambda \to \infty$, i.e. the integration limits for each v_k will become $\pm\infty$ as $\lambda \to \infty$. Thus by noticing that $m(\epsilon) \to 0$, $M(\epsilon) \to 0$ $(\lambda \to \infty)$ and using Lemma 1, we easily obtain [cf. (**6**) 626]

$$\lim_{\lambda\to\infty}\sqrt{|B|}\bar{J}(\lambda) = \lim_{\lambda\to\infty}\sqrt{|B|}\underline{J}(\lambda) = \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}vv')\,dv_1...dv_n = (2\pi)^{\frac{1}{2}n}.$$

From (27), (23), and (8) we may therefore infer that

$$I_1(\lambda) \sim \exp[f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)]|B|^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}n}$$
$$\sim \exp[f(\xi;\lambda)]\{(2\pi)^n/H_n[-f(\xi;\lambda)]\}^{\frac{1}{2}}. \tag{31}$$

Finally we have to verify condition (III). We have seen that the hypersurface $u = f(x;\lambda)$ (for fixed $\lambda > M$) in the system $(x_1,..., x_n; u)$, i.e. Euclidean $(n+1)$-space, is concave downward. Hence we see that in the closed region $R-U$ the function $f(x;\lambda)$ can take its maximum

value only at the boundary point of U. Let $\tilde{U}$ denote the boundary of U and let $||R||$ be the volume of R. Clearly we have

$$I_2(\lambda) \leqslant ||R|| \max\{\exp[f(x;\lambda)],\, x \in \tilde{U}\} = ||R|| \mu(\lambda), \tag{32}$$

where $\mu(\lambda)$ denotes the maximum or the least upper bound of $\exp[f(x;\lambda)]$ as x varies over $\tilde{U}$. Recall that U is a hypercube of side-length $2\epsilon(\lambda)$ containing the point $(\phi_1(\lambda),..., \phi_n(\lambda))$ as its centre, so that every point of $\tilde{U}$ may be written in the form

$$x^0 = (x_1^0,..., x_n^0) = (\phi_1(\lambda)+r_1\,\epsilon(\lambda),..., \phi_n(\lambda)+r_n\,\epsilon(\lambda)),$$

where $|r_i| \leqslant 1$ $(i = 1,..., n)$ and $|r_k| = 1$ for at least one k $(1 \leqslant k \leqslant n)$. Now suppose that x^0 is a point at which $\exp[f(x;\lambda)]$ takes the value $\mu(\lambda)$. We may write [cf. (23)]

$$\mu(\lambda) = \exp\Big[f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)+\tfrac{1}{2}\epsilon^2 \sum_{i=1}^{n} \sum_{k=1}^{n} f_{ik}(X_1,..., X_n;\lambda) r_i r_k\Big]. \tag{33}$$

An application of Lemma 5 gives, when $\lambda \to \infty$,

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (-) f_{ik}(X_1,..., X_n;\lambda) r_i r_k \asymp \sum_{i=1}^{n} \sum_{k=1}^{n} (-) f_{ik}(\phi_1(\lambda),..., \phi_n(\lambda);\lambda) r_i r_k. \tag{34}$$

Clearly $$(r_1^2+...+r_n^2)^{\frac{1}{2}} = \rho \geqslant 1.$$

We define $r_k/\rho = \cos\theta_k$ so that the right-hand side of (34) can be normalized and written as

$$-\rho^2(zHz') = -\rho^2\left(\frac{\partial^2 u}{\partial \Gamma_\theta^2}\right), \qquad z = (\cos\theta_1,..., \cos\theta_n), \qquad H = [f_{ik}]. \tag{35}$$

Hence by the same reasoning as already used in the proof of Lemma 4 [see (12), (13), (14), (15), (16)] and from (34), (35) we may infer that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (-) f_{ik}(X;\lambda) r_i r_k \asymp H_n[-f]/H_{n-1}[-g]+O(H_1[-f]) \asymp H_1[-f], \tag{36}$$

where $O(H_1[-f])$ or $O(-f_{11})$ is a non-negative term implied by (12), (13), and the fact that $H_s^{(k)} = O\{(-f_{11})^k\}$. Thus, by (31), (32), (33), (36), (29) we have

$$\begin{aligned} 0 < \frac{I_2(\lambda)}{I_1(\lambda)} &< K_1 \frac{||R||.|B|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}n}} \exp\Big[\tfrac{1}{2}\epsilon(\lambda)^2 \sum_{i=1}^{n} \sum_{k=1}^{n} f_{ik}(X;\lambda) r_i r_k\Big] \\ &< K_2 |B|^{\frac{1}{2}} \exp[K_3\,\epsilon(\lambda)^2 f_{11}(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)] \\ &= K_2 |B|^{\frac{1}{2}} \exp\{-K_3 K_0^2 \log H_n[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)]\} \\ &= K_2 |B|^{\frac{1}{2}} \{H_n[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)]\}^{-K_0^2 K_3} \\ &= K_2 \{H_n[-f(\phi_1(\lambda),..., \phi_n(\lambda);\lambda)]\}^{\frac{1}{2}-K_0^2 K_3}, \end{aligned} \tag{37}$$

where K_1, K_2, K_3 are certain positive numbers. Since the positive constant K_0 of (29) can be chosen freely, it may always be assumed that $K_0^2 K_3 > \frac{1}{2}$. Hence the expression (37) leads at once to the conclusion that

$$\lim\{I_2(\lambda)/I_1(\lambda)\} = 0$$

as $\lambda \to \infty$, and Theorem 1 is completely proved.

It is clear that the domain of integration R in the formula (1) can be replaced by D, if condition (v) is added. For we need only replace $||R||$ by $||D||$ in (32).

For the proof of Theorem 2, it suffices to consider the integral

$$I(\lambda) = \int \underset{R}{\dots} \int [\phi(x) - \phi(\xi)] \exp[f(x;\lambda)]\, dx_1 \dots dx_n,$$

where R is an arbitrary neighbourhood of $x = \xi$. Since $\phi(x)$ is continuous at $x = \xi$, the difference $|\phi(x) - \phi(\xi)|$ can be made as small as we please provided that R is sufficiently small. Thus by means of Theorem 1 and the first mean-value theorem for integrals we easily prove that

$$I(\lambda) = o\,(\exp[f(\xi;\lambda)]\{(2\pi)^n / H_n[-f(\xi;\lambda)]\}^{\frac{1}{2}}) \quad (\lambda \to \infty).$$

6. Proof of Theorem 3

I now sketch a proof of Theorem 3. Analysing the proof of Theorem 1, one easily finds that condition (iv) is merely introduced to meet the requirement

$$H_k[-f]/H_{k-1}[-f] \to \infty, \qquad H_n[-f]/H_{n-1}[-g] \to \infty, \tag{38}$$

as $(x;\lambda) \to (\xi;\infty)$ [see (14), (17), (30), (36), etc.], so that we have (17), (19), and an $\epsilon(\lambda)$-function satisfying conditions (I), (II), (III).

Now condition (iv a) implies that

$$\log H_n[-f] \to \infty, \qquad H_{n-1}[-g] = O(h^{n-1}), \quad H_k[-f]/H_{k-1}[-f] \to \infty,$$
$$H_n[-f]/(h(x;\lambda))^{n-1} \to \infty$$

as $(x;\lambda) \to (\xi;\infty)$, so that (38) is obviously satisfied. In the inequalities (16), (17) we need only replace $f_{11}(\xi;\lambda)$ by $h(\xi;\lambda)$, so that the proof of Lemma 4 is still valid. I am now going to define an $\epsilon(\lambda)$-function such that (I), (II), (III) are all satisfied. For every fixed $\lambda > N$ and $x \in R$, define

$$\mu(x;\lambda) = \min\left\{\frac{H_1[-f]}{H_0[-f]}, \frac{H_2[-f]}{H_1[-f]}, \dots, \frac{H_n[-f]}{H_{n-1}[-f]}, \frac{H_n[-f]}{|h(x;\lambda)|^{n-1}}\right\}, \tag{39}$$

where $H_0[-f] = 1$. Thus an $\epsilon(\lambda)$-function may simply be defined as follows:

$$\epsilon(\lambda) = K_0\left(\frac{\log H_n[-f(\phi_1(\lambda),\dots,\phi_n(\lambda);\lambda)]}{\mu(\phi_1(\lambda),\dots,\phi_n(\lambda);\lambda)}\right)^{\frac{1}{2}}. \tag{40}$$

Clearly the above $\epsilon(\lambda)$ function satisfies the requirement (I). Moreover, by (iv a), (39), and (40) we easily find that

$$v_k^2 = \epsilon(\lambda)^2 H_k[-f]/H_{k-1}[-f] \to \infty$$

[cf. (30)], so that (II) is also satisfied. To verify (III), let us notice that (36) can be replaced by

$$\sum_{i=1}^{n}\sum_{k=1}^{n}(-)f_{ik}(X;\lambda)r_i r_k \asymp H_n[-f]/H_{n-1}[-g]+O(|h|^{2n}) \quad ((x;\lambda)\to(\xi;\infty)), \tag{36$'$}$$

where

$$X = (X_1,..., X_n), \qquad X_k = X_k(x), \qquad H_{n-1}[-g(y;\lambda)] = O(|h(x;\lambda)|^{n-1})$$

and $O(|h(x;\lambda)|^{2n})$ is a non-negative term implied by (12) and (13), using (iv a). Clearly by (39), (36)$'$, and by the continuity of f and h we have

$$\mu(\phi(\lambda);\lambda) \leqslant \frac{H_n[-f(\phi(\lambda);\lambda)]}{|h(\phi(\lambda);\lambda)|^{n-1}} = O\Big\{\sum_{i=1}^{n}\sum_{k=1}^{n}(-)f_{ik}(X;\lambda)r_i r_k\Big\},$$

where $\phi(\lambda) = (\phi_1(\lambda),..., \phi_n(\lambda))$. Thus by the same procedure as previously used in the derivation of (37) with slight modifications and notational changes [e.g. $f_{11}(\phi(\lambda);\lambda)$ must be replaced by $-\mu(\phi(\lambda);\lambda)$; K_3 (> 0) may depend on λ and tend to infinity† with λ] we easily obtain

$$\lim\{I_2(\lambda)/I_1(\lambda)\} = 0 \quad (\lambda\to\infty).$$

Hence in conclusion Theorem 3 is true.

7. Determination of ξ and the boundary-point case

From Lemma 4 it is clear that the existence of $\xi = (\xi_1,..., \xi_n)$ is unique. Since $f_k(x;\lambda)$ are continuous functions and $\phi_k(\lambda)\to\xi_k$ as $\lambda\to\infty$, we see that the approximate values of $\xi_1,..., \xi_n$ are obtainable by solving simultaneous equations $f_k(x;\lambda) = 0$ for large λ. For some special cases, $\phi_k(\lambda)$ can be explicitly determined so that $(\xi_1,..., \xi_n)$ may sometimes be easily obtained by letting $\lambda\to\infty$ in $\phi_k(\lambda)$.

The other special case is as follows. Set $\lambda = 1/t$ so that $f(x;\lambda) = g(x;t)$ and $f_k(x;\lambda) = g_k(x;t)$, where $t > 0$. By (5), (8), or condition (i) we easily see that $\xi = (\xi_1,..., \xi_n)$ satisfies the equations $g_k(\xi_1,..., \xi_n; 0+) = 0$. Thus, in particular, if $g_k(x;t)$ are continuous at $t = 0$, the point $(\xi_1,..., \xi_n)$ may be directly obtained by solving $g_k(x_1,..., x_n; 0) = 0$, provided that the solution of the latter system of equations is unique.

Let $\tilde{R}$ be the boundary of R. In a previous paper (9) it was shown that, if ξ is a boundary point of R and if $\tilde{R}$ possesses a continuously turning tangent plane near the ordinary point ξ, then the asymptotic value of the multiple integral of $\exp[\lambda f(x_1,..., x_n)]$ taken over the closed

† In such a case it is obvious that the expression (37) will tend to 0 with $1/\lambda$ for every positive K_0.

domain R equals just half the value for the case where ξ is an interior point. Such a conclusion cannot be extended to the general case here treated. In fact, if $d(\lambda)$ denotes the shortest distance of the variable point $(\phi_1(\lambda),..., \phi_n(\lambda))$ from the boundary $\tilde{R}$, and if $\epsilon(\lambda)$ is defined by (29) or (40) such that, as $\lambda \to \infty$,

$$\epsilon(\lambda) = o\{d(\lambda)\},$$

it is then easy to see that the validity of the asymptotic formula (1) is independent of the position of ξ. For, if ξ belongs to $\tilde{R}$, it will always lie outside our variable neighbourhood (e.g. hypercube) U_ϵ of $(\phi_1(\lambda),..., \phi_n(\lambda))$ for large λ, although $(\phi_1(\lambda),..., \phi_n(\lambda)) \to \xi$ when $\lambda \to \infty$. Since our asymptotic evaluation of the multiple integral merely depends on the choice of U_ϵ within R and without referring to ξ [see § 6], it is clear that the asymptotic formula (1) is still true provided that (i), (ii), (iii), (iv)—or (iv a)—are fulfilled.

Finally it may be worthy of mention that the 1-dimensional case of Theorem 1 can be extended to the case in which the range of integration is infinite, without adding new conditions. In fact, by a modification of Haviland's method (**6**) it has been proved previously (**10**) that

$$\int_{-\infty}^{\infty} \exp[f(x;\lambda)]\,dx \sim \exp[f(\xi;\lambda)]\{-2\pi/f''_{xx}(\xi;\lambda)\}^{\frac{1}{2}},$$

whenever conditions (i), (ii), (iii) of Theorem 1 are satisfied with $n = 1$, $R = (-\infty, \infty)$. Note that condition (iv) is automatically satisfied when $n = 1$. Moreover, the 2-dimensional case may compared with the sesults contained in (11) and (12).

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ON DIAGONABLE AND NORMAL MATRICES

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Introduction

If we consider the set of all $n \times n$ matrices over any given (algebraically closed) field, then it is natural to regard those having *diagonal* classical canonical forms (i.e. the *diagonable* matrices) as forming an important subset; for any given square matrix A will certainly be reducible to diagonal form by a similarity transformation $P^{-1}AP$ if all its eigenvalues are *distinct*, while the condition for this can be expressed as a single algebraic inequality involving the elements of A (so that we may say that almost every square matrix is diagonable).

Further, we shall see that diagonable matrices have a variety of interesting special properties, while, in the *complex* field, the set of diagonable matrices includes all *normal* matrices (a given square matrix A being called normal if it commutes with its conjugate transpose A^*); again, it will appear that the set of normal matrices (which, in turn, obviously includes all unitary, hermitian, and skew-hermitian matrices) has several simple characteristic properties.

Some of these matters had been investigated even before the fundamental axioms of matrix algebra had been explicitly formulated, and the literature is extensive; considerations of space preclude a full historical introduction, but some historical remarks are appended to the various theorems. Many of the results are already known, as disconnected facts, but there appears to have been no previous attempt to give a systematic and unified treatment of the subject; I have tried to present the material in as general a form as possible (i.e. with the minimum of reference to the concept of a matrix as a formal array).

Diagonable matrices

Theorem 1. *If A is any given $n \times n$ matrix, then each of the following conditions is necessary and sufficient for the diagonability of A:*

(i) *For every n-vector ξ, and every scalar λ,*

$$(A-\lambda I)^2\xi = 0 \quad \textit{implies} \quad (A-\lambda I)\xi = 0;$$

(ii) *Given any eigenvector ξ of A, with eigenvalue α, say, the vector equation $(A-\alpha I)\eta = \xi$ has no solution η;*

(iii) *There is a scalar polynomial* $\psi(x)$, *with no multiple zero, such that* $\psi(\boldsymbol{A}) = \boldsymbol{0}$;

(iv) *There is an integer* r, *scalars* $\lambda_1,\ldots, \lambda_r$, *and matrices* $\boldsymbol{E}_1,\ldots, \boldsymbol{E}_r$ *such that*

$$(a)\ \sum_{j=1}^{r} \lambda_j \boldsymbol{E}_j = \boldsymbol{A},$$

$$(b)\quad \sum_{j=1}^{r} \boldsymbol{E}_j = \boldsymbol{I},$$

$$(c)\quad \boldsymbol{E}_j \boldsymbol{E}_k = \boldsymbol{0} \quad (j \neq k);$$

(v) $\boldsymbol{A}$ *has* n *linearly independent eigenvectors;*

(vi) *Given any eigenvalue* α_k *of* $\boldsymbol{A}$, *with multiplicity* m_k, *say, then* $\boldsymbol{A}-\alpha_k \boldsymbol{I}$ *has rank* $n-m_k$, *i.e. the space of eigenvectors of* $\boldsymbol{A}$ *with eigenvalue* α_k *has dimension* m_k-1.

Further, if any of these conditions holds, then, in (iii), *if* $\alpha_1,\ldots, \alpha_p$ *are the distinct eigenvalues of* $\boldsymbol{A}$,

$$\phi_0(x) \equiv \prod_{k=1}^{p} (x-\alpha_k)$$

must divide $\psi(x)$, *and, in fact,* $\phi_0(\boldsymbol{A}) = \boldsymbol{0}$.

Proof. I shall refer to the hypothesis that $\boldsymbol{A}$ is diagonable as D; I first prove (cyclically) the equivalence of D, (i), (ii), (iii), (iv), and (v), and then prove the equivalence of (v), (vi) directly.

D *implies* (i). If D holds, then let $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ be diagonal, and, given any n-vector $\boldsymbol{\xi}$, write $\boldsymbol{P}^{-1}\boldsymbol{\xi} = \boldsymbol{\eta}$.

Then, if $\quad (\boldsymbol{A}-\lambda \boldsymbol{I})^2\boldsymbol{\xi} = \boldsymbol{0},$

we have $\quad (\boldsymbol{D}-\lambda \boldsymbol{I})^2\boldsymbol{\eta} = \boldsymbol{0},$

so that $\quad (\boldsymbol{D}-\lambda \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{0},$

whence $\quad (\boldsymbol{A}-\lambda \boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$, as required.

(i) *implies* (ii). If $\boldsymbol{A}\boldsymbol{\xi} = \alpha\boldsymbol{\xi}$, then $(\boldsymbol{A}-\alpha \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ implies that

$$(\boldsymbol{A}-\alpha \boldsymbol{I})^2\boldsymbol{\eta} = (\boldsymbol{A}-\alpha \boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0},$$

whence, if (i) holds, then $\boldsymbol{\xi} = (\boldsymbol{A}-\alpha \boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{0}$, and is therefore *not* an eigenvector of $\boldsymbol{A}$, i.e. (ii) is proved.

(ii) *implies* (iii). Let the minimum polynomial of $\boldsymbol{A}$ be

$$\phi(x) = \prod_{k=1}^{p} (x-\alpha_k)^{s_k},$$

and suppose that there is an $s_k > 1$, say for $k = j$.

Then, defining the polynomial $f(x)$ by the relation

$$\phi(x) \equiv (x-\alpha_j)^2 f(x),$$

we have
$$(\boldsymbol{A}-\alpha_j \boldsymbol{I})f(\boldsymbol{A}) \neq \boldsymbol{0},$$

so there is a vector $\boldsymbol{\zeta}$ such that

$$\boldsymbol{\xi} = (\boldsymbol{A}-\alpha_j \boldsymbol{I})f(\boldsymbol{A})\boldsymbol{\zeta} \neq \boldsymbol{0}.$$

But then
$$(\boldsymbol{A}-\alpha_j \boldsymbol{I})\boldsymbol{\xi} = \phi(\boldsymbol{A})\boldsymbol{\zeta} = \boldsymbol{0},$$

so that $\boldsymbol{\xi}$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue α_j, contrary to (ii); hence, if (ii) holds, then every $s_k \leqslant 1$, i.e. $\phi(x)$ has all its roots simple, so that (iii) holds.

(iii) *implies* (iv). If (iii) holds, then let $\psi(x)$ have zeros $\lambda_1,\ldots, \lambda_r$, and define polynomials $\psi_j(x)$ by the identities

$$\psi(x) \equiv (x-\lambda_j)\psi_j(x),$$

so that
$$\psi_j(\lambda_k) = 0 \quad \text{if} \quad j \neq k,$$

while
$$\psi_j(\lambda_j) \neq 0 \quad (j = 1, 2,\ldots, r).$$

Next, defining
$$g(x) = \sum_{j=1}^{r} \frac{\psi_j(x)}{\psi_j(\lambda_j)} - 1,$$

we have
$$g(\lambda_k) = 0 \quad (k = 1, 2,\ldots, r),$$

whence, since clearly $g(x)$ has degree $r-1$ at most, we deduce that $g(x) \equiv 0$, and therefore, taking

$$\boldsymbol{E}_j = \{\psi_j(\lambda_j)\}^{-1}\psi_j(\boldsymbol{A}),$$

we have
$$\sum_{j=1}^{r} \boldsymbol{E}_j = \boldsymbol{I}.$$

Further,
$$\lambda_j \psi_j(A) = \boldsymbol{A}\psi_j(\boldsymbol{A}) - \psi(\boldsymbol{A}) = \boldsymbol{A}\psi_j(\boldsymbol{A}),$$

whence
$$\sum_{j=1}^{r} \lambda_j \boldsymbol{E}_j = \boldsymbol{A} \sum_{j=1}^{r} \boldsymbol{E}_j = \boldsymbol{A};$$

also, since $\psi_j(x)\psi_k(x)$ is divisible by $\psi(x)$ whenever $j \neq k$, therefore

$$\boldsymbol{E}_j \boldsymbol{E}_k = \boldsymbol{0} \quad \text{whenever} \quad j \neq k.$$

Thus, given (iii), we have constructed scalars $\lambda_1,\ldots, \lambda_r$, and a set of matrices $\boldsymbol{E}_1,\ldots, \boldsymbol{E}_r$ satisfying (*a*), (*b*), (*c*) of (iv), as required.

(iv) *implies* (v). If (iv) holds, then, given any n-vector $\boldsymbol{\zeta}$, we have

$$\boldsymbol{\zeta} = \boldsymbol{I\zeta} = \sum_{j=1}^{r} \boldsymbol{E}_j \boldsymbol{\zeta},$$

while also
$$\boldsymbol{A}\boldsymbol{E}_j \boldsymbol{\zeta} = \Big(\sum_j \lambda_j \boldsymbol{E}_j\Big)\boldsymbol{E}_j \boldsymbol{\zeta} = \lambda_j \boldsymbol{E}_j^2 \boldsymbol{\zeta} = \lambda_j \boldsymbol{E}_j\Big(\sum_j \boldsymbol{E}_j\Big)\boldsymbol{\zeta}$$
$$= \lambda_j \boldsymbol{E}_j \boldsymbol{\zeta},$$

so that any vector $\boldsymbol{\zeta}$ can be expressed as a linear combination of eigenvectors of $\boldsymbol{A}$, i.e. (v) holds.

(v) *implies* D. If $\boldsymbol{A}$ has n linearly independent eigenvectors, then these can be taken as the columns of a non-singular matrix, $\boldsymbol{P}$, say, and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is then clearly diagonal (the diagonal elements being, in order, the eigenvalues of $\boldsymbol{A}$ for the corresponding columns of $\boldsymbol{P}$).

(vi) *is equivalent to* (v). If (vi) holds, then, given any eigenvalue α_k of $\boldsymbol{A}$, there are just $n-(n-m_k) = m_k$ linearly independent solutions $\boldsymbol{\xi}$ of the vector equation $(\boldsymbol{A}-\alpha_k \boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$; hence, since there can be no non-trivial relation between eigenvectors of $\boldsymbol{A}$ all corresponding to different eigenvalues, we can construct a set of $m_1+m_2+\ldots+m_p = n$ linearly independent eigenvectors of $\boldsymbol{A}$, i.e. (v) holds.

Conversely, if (v) holds, then, by what we have already proved, D holds, whence (vi) is immediate, since rank is unaltered by multiplication by a non-singular matrix.

We have now proved the necessity and sufficiency of each of the stated conditions for the diagonability of $\boldsymbol{A}$.

Finally, if (iii) holds, and if $\boldsymbol{\xi}$ is any eigenvector corresponding to a given eigenvalue α_k of $\boldsymbol{A}$, then we have

$$\psi(\alpha_k)\boldsymbol{\xi} = \psi(\boldsymbol{A})\boldsymbol{\xi} = 0,$$

so that $$\psi(\alpha_k) = 0 \quad (k = 1, 2,\ldots, p),$$

and therefore $\phi_0(x)$ divides $\psi(x)$, whence, in fact, $\phi_0(\boldsymbol{A}) = \boldsymbol{0}$, since $\boldsymbol{A}-\alpha \boldsymbol{I}$ is non-singular whenever α is not an eigenvalue of $\boldsymbol{A}$. Thus the proof of the theorem is completed.

The necessity of each of the conditions of the theorem is, of course, obvious, while also each of D, (i), (ii), (iii) is easily deduced directly from any given one of (i), (ii), (iii), (iv), (v), so there is a very wide choice of methods of proof. However, any proof is inevitably somewhat tedious, and (unless we resort to the general theory of canonical forms) it does not seem possible to obtain all the results of the theorem appreciably more shortly than as above.

The equivalence of condition (v) to diagonability is trivial, while that of (iii), (iv), (vi) is well known. The remaining conditions (i), (ii), though probably the most powerful results available for proving the diagonability of a given class of matrices, have, apparently, never been stated explicitly; however, (i) has been used by Dirac [(**3**), 28–34], and (ii) by Baker (**1**) and Wedderburn [(**10**), 92].

Corollary 1. *If* $\boldsymbol{A}$ *is any given square matrix, and* $\psi(x)$ *any given (scalar) polynomial with no multiple zero, then a necessary and sufficient*

condition that $\psi(\boldsymbol{A}) = \mathbf{0}$ *is that* $\boldsymbol{A}$ *be diagonable and that, for every eigenvalue* α *of* $\boldsymbol{A}$, $\psi(\alpha) = 0$.

Proof. For, if $\psi(\boldsymbol{A}) = \mathbf{0}$, then, by the sufficiency of (iii), $\boldsymbol{A}$ is diagonable, while, since $\phi_0(x)$ must divide $\psi(x)$, therefore $\psi(\alpha) = 0$ for each eigenvalue α of $\boldsymbol{A}$.

Conversely, if $\psi(\alpha) = 0$ for each eigenvalue α of $\boldsymbol{A}$, then $\psi(x)$ is divisible by $\phi_0(x)$; then, if $\boldsymbol{A}$ is diagonable, so that $\phi_0(\boldsymbol{A}) = \mathbf{0}$, it follows that $\psi(\boldsymbol{A}) = \mathbf{0}$, as required.

In particular, if m is any given positive integer, then we see that a given matrix $\boldsymbol{A}$ will satisfy the periodic equation $\boldsymbol{A}^m = \boldsymbol{I}$ (or the more general equation $\boldsymbol{A}^{m+1} = \boldsymbol{A}$) if and only if $\boldsymbol{A}$ is diagonable, and each of its eigenvalues satisfies the same equation.

COROLLARY 2. *If* $\boldsymbol{A}$ *is any given square matrix (with distinct eigenvalues* $\alpha_1,\ldots,\alpha_p$, *say), then* $\boldsymbol{A}$ *is diagonable if and only if there are matrices* $\boldsymbol{E}_1,\ldots,\boldsymbol{E}_p$ *such that*

$$(a) \quad \boldsymbol{A} = \sum_{k=1}^{p} \alpha_k \boldsymbol{E}_k,$$

$$(b) \quad \sum_{k=1}^{p} \boldsymbol{E}_k = \boldsymbol{I},$$

$$(c) \quad \boldsymbol{E}_j \boldsymbol{E}_k = \mathbf{0} \quad (j \neq k);$$

if this is the case, then the matrix $\boldsymbol{E}_k$ *corresponding to any given* α_k *is uniquely determined, and, in addition,*

$$(d) \quad \boldsymbol{E}_k^2 = \boldsymbol{E}_k \quad (k = 1, 2,\ldots, p);$$

(*e*) *each* $\boldsymbol{E}_k$ *can be expressed as a polynomial in* $\boldsymbol{A}$;

(*f*) *the columns of any given* $\boldsymbol{E}_k$ *span the space of eigenvectors of* $\boldsymbol{A}$ *with eigenvalue* α_k.

It then follows from (*a*), (*e*) *that any given matrix* $\boldsymbol{B}$ *will commute with* $\boldsymbol{A}$ *if and only if*

$$\boldsymbol{B}\boldsymbol{E}_k = \boldsymbol{E}_k \boldsymbol{B} \quad (k = 1, 2,\ldots, p).$$

Proof. Taking $\psi(x) = \phi_0(x)$ in the construction used in the theorem to prove that (iii) implies (iv), we see at once that, if $\boldsymbol{A}$ is diagonable, then matrices $\boldsymbol{E}_1,\ldots,\boldsymbol{E}_p$ can be found to satisfy (*a*), (*b*), (*c*);† conversely, if this is the case, then, by the theorem, $\boldsymbol{A}$ is certainly diagonable.

† For an alternative proof of this we may observe that, in any given general representation of the type (iv), each λ_j for which $\boldsymbol{E}_j \neq \mathbf{0}$ must in fact be an eigenvalue of $\boldsymbol{A}$, and that terms corresponding to equal values of λ_j can be combined without affecting any of the properties (*a*), (*b*), (*c*).

Then, by (b), (c), we find

$$\boldsymbol{E}_k = \boldsymbol{E}_k\Big(\sum_j \boldsymbol{E}_j\Big) = \boldsymbol{E}_k^2, \quad \text{proving } (d),$$

whence $$\boldsymbol{A}\boldsymbol{E}_k = \alpha_k \boldsymbol{E}_k = \boldsymbol{E}_k \boldsymbol{A} \quad (k = 1, 2,..., p).$$

If also $$\boldsymbol{A} = \sum_{j=1}^{p} \alpha_j \boldsymbol{F}_j,$$

where the $\boldsymbol{F}_j$ satisfy (b), (c), and hence also (d), then we have

$$\begin{aligned}(\alpha_j - \alpha_k)\boldsymbol{E}_k \boldsymbol{F}_j &= \boldsymbol{E}_k(\alpha_j \boldsymbol{F}_j) - (\alpha_k \boldsymbol{E}_k)\boldsymbol{F}_j \\ &= (\boldsymbol{E}_k \boldsymbol{A} - \boldsymbol{A}\boldsymbol{E}_k)\boldsymbol{F}_j = \boldsymbol{0},\end{aligned}$$

whence, if $j \neq k$, then $\boldsymbol{E}_k \boldsymbol{F}_j = \boldsymbol{0}$, and similarly $\boldsymbol{F}_j \boldsymbol{E}_k = \boldsymbol{0}$.

Then, by (b), it follows that

$$\boldsymbol{F}_j = \boldsymbol{F}_j\Big(\sum_k \boldsymbol{E}_k\Big) = \boldsymbol{F}_j \boldsymbol{E}_j = \Big(\sum_k \boldsymbol{F}_k\Big)\boldsymbol{E}_j = \boldsymbol{E}_j \quad (j = 1, 2,..., p),$$

i.e. the representation (a), subject to (b), (c), is *unique*; also, by the construction used in the theorem to prove the existence of $\boldsymbol{E}_1,..., \boldsymbol{E}_r$, ($e$) is immediate, so we have now only to prove (f).

Since $\boldsymbol{A}\boldsymbol{E}_j = \alpha_j \boldsymbol{E}_j$, the columns of $\boldsymbol{E}_j$ obviously span a *subspace* of the space of eigenvectors of $\boldsymbol{A}$ with eigenvalue α_j. Also, if $\boldsymbol{\zeta}$ is any eigenvector of $\boldsymbol{A}$ with eigenvalue α_j, then we have

$$(\alpha_j - \alpha_k)\boldsymbol{E}_k \boldsymbol{\zeta} = \boldsymbol{E}_k . \boldsymbol{A}\boldsymbol{\zeta} - \boldsymbol{A}\boldsymbol{E}_k . \boldsymbol{\zeta} = \boldsymbol{0},$$

so that $\boldsymbol{E}_k \boldsymbol{\zeta} = \boldsymbol{0}$ whenever $k \neq j$, and therefore, by (b),

$$\boldsymbol{\zeta} = \sum_{k=1}^{p} \boldsymbol{E}_k \boldsymbol{\zeta} = \boldsymbol{E}_j \boldsymbol{\zeta},$$

i.e. *any* eigenvector of $\boldsymbol{A}$ with eigenvalue α_j can be represented as a linear combination of the columns of the matrix $\boldsymbol{E}_j$ corresponding to α_j, so (f) follows, and the proof of the corollary is completed.

The matrices $\boldsymbol{E}_k$ of Corollary 2 are known as the *principal idempotent elements* (or *Frobenius covariants*) of $\boldsymbol{A}$, and their definition can be extended† so as to apply to *any* given square matrix.

THEOREM 2.‡ *If $\boldsymbol{A}_1,..., \boldsymbol{A}_m$ is any given set of $n \times n$ matrices, then the following statements are equivalent:*

(i) *each $\boldsymbol{A}_i$ is diagonable, and $\boldsymbol{A}_i \boldsymbol{A}_j = \boldsymbol{A}_j \boldsymbol{A}_i$ ($i, j = 1, 2,..., m$);*

(ii) *each of $\boldsymbol{A}_1,..., \boldsymbol{A}_m$ can be expressed as a polynomial in the same diagonable matrix ($\boldsymbol{A}$, say);*

† See Wedderburn (**10**), 25–30.

‡ This has been previously stated by Dungey, Gruenberg, and myself (**4**); however, the proof given here is new, and produces a slightly stronger form of the result. Cf. Cherubino (**2**), Halmos (**5**), 141.

(iii) $A_1,\ldots, A_m$ *can be simultaneously reduced to diagonal form by a similarity transformation* $P^{-1}A_iP$ (*i.e.* $A_1,\ldots, A_m$ *have* n *common linearly independent eigenvectors*).

Proof. Obviously (ii) implies (iii), and (iii) implies (i); it will therefore be sufficient to prove that (i) implies (ii).

Let A_i have $\alpha_1^{(i)},\ldots, \alpha_{p_i}^{(i)}$ as its distinct eigenvalues (with suitable multiplicities) and corresponding idempotent elements $E_{k_i}^{(i)}$, and let $\gamma_{k_1,\ldots,k_m}$ be any set of distinct scalars ($k_i = 1, 2,\ldots, p_i$; $i = 1, 2,\ldots, m$).

Then, by Theorem 1 $\left(\text{with } r = \prod_{i=1}^{m} p_i\right)$, the matrix

$$A = \sum_{k_1,\ldots,k_m} \gamma_{k_1,\ldots,k_m} E_{k_1}^{(1)}\ldots E_{k_m}^{(m)}$$

is diagonable, while, choosing polynomials $f_j(x)$ such that

$$f_j(\gamma_{k_1,\ldots,k_m}) = \alpha_{k_j}^{(j)} \quad (k_j = 1, 2,\ldots, p_j;\ j = 1, 2,\ldots, m),$$

we have
$$f_j(A) = f_j\Big(\sum_{k_1,\ldots,k_m} \gamma_{k_1,\ldots,k_m} E_{k_1}^{(1)}\ldots E_{k_m}^{(m)}\Big)$$
$$= \sum_{k_1,\ldots,k_m} f_j(\gamma_{k_1,\ldots,k_m}) E_{k_1}^{(1)}\ldots E_{k_m}^{(m)}$$

since the A's commute, by Corollary 2 (c), (d); hence, using (a), (b), we find
$$f_j(A) = \sum_{k_1,\ldots,k_m} \alpha_{k_j}^{(j)} E_{k_1}^{(1)}\ldots E_{k_m}^{(m)} = A_j,$$
as required.

(Incidentally, by Corollary 2 (e), the matrix A we have constructed is a polynomial in $A_1,\ldots, A_m$, so the eigenvectors of this A are precisely the *common* eigenvectors of $A_1,\ldots, A_m$.)

Normal matrices

We now leave the class of general diagonable matrices, and discuss the subclass of normal matrices (over the complex field); I shall first prove the equivalence of the defining property of normality (i.e. $AA^* = A^*A$) to some other useful criteria.

THEOREM 3. *If* $A = (a_{ij})$ *is any given* $n \times n$ *matrix, then each of the following conditions is necessary and sufficient for the normality of* A:

(i) A *is diagonable, and each of its idempotent elements is hermitian*;

(ii) A^* *can be represented as a polynomial in* A;

(iii) A *can be reduced to diagonal form by a unitary similarity transformation*;

(iv)
$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{k=1}^{n} |\alpha_k|^2,$$

where $\alpha_1,\ldots, \alpha_n$ *is the set of all eigenvalues of* A (*counted with their multiplicities*).

Proof. Denoting by N the hypothesis that A is normal, I first prove (cyclically) the equivalence of N, (i), (ii), and (iii), and then prove directly that (iv) is equivalent to (iii).

N *implies* (i). If A is normal, and $\boldsymbol{\xi}$ is any eigenvector of A, say $A\boldsymbol{\xi} = \alpha\boldsymbol{\xi}$, then we have

$$(A^*\boldsymbol{\xi}-\bar{\alpha}\boldsymbol{\xi})^*(A^*\boldsymbol{\xi}-\bar{\alpha}\boldsymbol{\xi}) = \boldsymbol{\xi}^*(A-\alpha I)(A^*-\bar{\alpha}I)\boldsymbol{\xi}$$
$$= \boldsymbol{\xi}^*(A^*-\bar{\alpha}I)(A-\alpha I)\boldsymbol{\xi} = 0,$$

so that $A^*\boldsymbol{\xi} = \bar{\alpha}\boldsymbol{\xi}$. Thus any relation of the form $(A-\alpha I)\boldsymbol{\eta} = \boldsymbol{\xi}$ would imply

$$\boldsymbol{\xi}^*\boldsymbol{\xi} = \boldsymbol{\eta}^*(A-\alpha I)^*\boldsymbol{\xi} = \boldsymbol{\eta}^*(A^*\boldsymbol{\xi}-\bar{\alpha}\boldsymbol{\xi}) = 0;$$

hence, by Theorem 1 (ii), A is diagonable; so we can write $A = \sum_{k=1}^{p} \alpha_k E_k$.†

Then $AE_k = \alpha_k E_k$; so, by what we have just proved, $A^*E_k = \bar{\alpha}_k E_k$, and therefore

$$\alpha_k E_k E_j^* = (E_k A)E_j^* = E_k(A^*E_j)^* = E_k(\bar{\alpha}_j E_j)^* = \alpha_j E_k E_j^*,$$

whence $E_k E_j^* = 0$ unless $\alpha_j = \alpha_k$, i.e. unless $j = k$; hence

$$E_k^* = E_k^*\Big(\sum_j E_j\Big) = E_k^* E_k = \Big(\sum_j E_j\Big)^* E_k = E_k \quad (k = 1, 2,..., p),$$

as required.

(i) *implies* (ii). If (i) holds, then, choosing a polynomial $f(x)$ such that

$$f(\alpha_k) = \bar{\alpha}_k \quad (k = 1, 2,..., p),$$

we have

$$f(A) = f\Big(\sum_k \alpha_k E_k\Big) = \sum_k f(\alpha_k)E_k = \sum_k \bar{\alpha}_k E_k = A^*,$$

as required.

(ii) *implies* (iii). It is easily proved by induction‡ that *any* given $n \times n$ matrix can be reduced to *triangular* form by a unitary similarity transformation, say $P^{-1}AP = T$, where every subdiagonal element of T is zero; then, if (ii) holds, say $A^* = f(A)$, we have

$$T^* = (P^{-1}AP)^* = P^{-1}A^*P = P^{-1}f(A)P = f(P^{-1}AP) = f(T);$$

but clearly any polynomial in T is also triangular, so that T is in fact *diagonal*, i.e. (iii) holds.

(iii) *implies* N. If there is a unitary matrix P such that $P^{-1}AP$ is diagonal, then $P^{-1}A^*P = (P^{-1}AP)^*$ is also diagonal; hence $P^{-1}AP$ commutes with $P^{-1}A^*P$, and N is immediate.

† We are here, of course, using $\alpha_1,..., \alpha_p$ to denote the set of all *distinct* eigenvalues of A (in contrast with (iv) of the statement of the present theorem where $\alpha_1,..., \alpha_n$ is the set of *all* the eigenvalues of A, possibly with repetitions). It is hoped that this slight inconsistency of notation will not lead to confusion.

‡ See, e.g., Turnbull and Aitken (8) 105.

(iv) *is equivalent to* (iii). As noted above, given any $n \times n$ matrix $A = (a_{ij})$, there is a unitary matrix P such that $T = P^{-1}AP$ is triangular; then

$$\begin{aligned}\sum_{i,j=1}^{n} |a_{ij}|^2 &= \operatorname{trace}(AA^*) = \operatorname{trace}(PTP^{-1}.PT^*P^{-1}) \\ &= \operatorname{trace}(PTT^*.P^{-1}) = \operatorname{trace}(TT^*) \\ &= \sum_{i,j=1}^{n} |t_{ij}|^2 \geqslant \sum_{i=1}^{n} |t_{ii}|^2 = \sum_{k=1}^{n} |\alpha_k|^2,\end{aligned}$$

with equality if and only if there is a unitary matrix P such that $P^{-1}AP$ is diagonal. This completes the proof of the theorem.

As with Theorem 1, this result can be derived in many ways; thus it is usual to prove the necessity of (iii) directly (by induction on n)—or we could first prove that every normal matrix is diagonable,† and then deduce the *unitary* diagonability by the familiar orthogonalization process. It is perhaps worth remarking that, in proving the necessity of (i), we have incidentally shown that another criterion for the normality of A is that A be diagonable, and that its idempotent elements satisfy $E_j^* E_k = 0$ whenever $j \neq k$; i.e. by Corollary 2 (f), a given square matrix A will be normal if and only if it is diagonable, and, in addition, eigenvectors of A corresponding to *different* eigenvalues are mutually *orthogonal* (in the complex sense).

As far as I am able to ascertain, the criteria of Theorem 3 are respectively due to Halmos, Williamson, Autonne, and Schur.

COROLLARY 1. *$AB = BA$ implies $A^*B = BA^*$ for every matrix B if and only if A is normal.*

Proof. If A is normal, then, by (ii) of Theorem 3 (or by (i) and Theorem 1, Corollary 2), obviously $AB = BA$ implies $A^*B = BA^*$.

Conversely, if $AB = BA$ implies $A^*B = BA^*$, the normality of A follows on taking $B = A$.

This corollary is apparently due to Halmos [(**5**) 141];‡ the first (i.e. non-trivial) part of the proof may also be deduced from von Neumann's identity (**9**)

$$||A^*B - BA^*||^2 - ||A^*B - BA^*||^2 \equiv \operatorname{trace}((AA^* - A^*A)(BB^* - B^*B)),$$

where $||A||^2 = \operatorname{trace}(AA^*) = \sum_{i,j=1}^{n} |a_{ij}|^2$. Further, using Kreis's result (**7**)§ that, if a matrix X commutes with every matrix which commutes

† e.g. as in the proof of the necessity of (i), or as in (**4**); I have also found two simple proofs depending respectively on (i), (iii) of Theorem 1.

‡ See also Wiegmann (**11**) for some interesting generalizations.

§ Or, more accessibly, Hamburger (**6**), Turnbull and Aitken (**8**) 149–50.

with a given matrix A, then X can be expressed as a polynomial in A, this identity also provides an alternative proof of the necessity of the condition (ii) in Theorem 3 (the sufficiency being trivial).

Another immediate corollary of Theorem 3 [cf. Turnbull and Aitken (**8**) 106–7] is the possibility of the simultaneous conjunctive transformation of a pair of hermitian matrices of which one is positive definite to diagonal and unit form respectively.

If each matrix A_i in Theorem 2 is normal, then clearly so is the matrix A we constructed, so that the reduction in (iii) can be taken to be unitary; in fact a great deal more can be proved [see (**4**)], but I shall here content myself with proving

COROLLARY 2. *If, in Theorem 2, there are scalars $\epsilon_1,..., \epsilon_m$ such that*

$$A_i^* = \epsilon_i A_i \quad (i = 1, 2,..., m),$$

then we can replace (i) *by the apparently weaker condition*

(i)′ *each matrix $A_i A_j - A_j A_i$ is nilpotent* $(i, j = 1, 2,..., m)$.

Proof. We have

$$(A_i A_j - A_j A_i)^* = A_j^* A_i^* - A_i^* A_j^* = -\epsilon_i \epsilon_j (A_i A_j - A_j A_i),$$

so each matrix $A_i A_j - A_j A_i$ is normal, and therefore diagonable; but a nilpotent diagonable matrix must clearly be the zero matrix, so (i), (i)′ are equivalent, as required.

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ON THE FINITE HILBERT TRANSFORMATION

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1. Hilbert's beautiful reciprocity formulae†

$$u(x) = \frac{1}{\pi}\int\limits_{-\infty}^{*\infty} \frac{v(y)}{y-x}\,dy, \qquad v(x) = -\frac{1}{\pi}\int\limits_{-\infty}^{*\infty} \frac{u(y)}{y-x}\,dy \tag{1}$$

generate, as is well known, an elegant and useful theory of *Hilbert's transformation*

$$f(x) = \frac{1}{\pi}\int\limits_{-\infty}^{*\infty} \frac{\phi(y)}{y-x}\,dy \equiv \mathscr{H}_x[\phi(y)], \tag{2}$$

which received a quite complete settlement in Titchmarsh's book on Fourier Integrals.‡ However, in some questions (e.g. in Aerodynamics) there arises naturally the *finite Hilbert transformation*

$$\mathscr{T}_x[\phi(y)] \equiv \frac{1}{\pi}\int\limits_{-1}^{*1} \frac{\phi(y)}{y-x}\,dy, \tag{3}$$

which has been so far less studied. Its properties cannot be always deduced from the corresponding properties of the 'infinite' transformation (2) by merely putting $\phi(x) \equiv 0$ outside the basic interval $(-1, 1)$.

The purpose of this paper is principally to give precise conditions for the *inversion* of the transformation (3), i.e. for the resolution of the well-known *airfoil equation*

$$\frac{1}{\pi}\int\limits_{-1}^{*1} \frac{\phi(y)}{y-x}\,dy = f(x) \quad (-1 < x < 1). \tag{4}$$

For this I use a kind of *convolution theorem* for the $\mathscr{T}$-transformation

$$\mathscr{T}\{\phi_1 \mathscr{T}[\phi_2] + \phi_2 \mathscr{T}[\phi_1]\} = \mathscr{T}[\phi_1]\mathscr{T}[\phi_2] - \phi_1\phi_2 \tag{5}$$

which—although valid (unchanged) even for the $\mathscr{H}$-transformation—

† In this paper an asterisk on the usual integral sign will indicate *Cauchy's principal value* of the integral of a certain function $f(x)$, which becomes infinite (of the first order) at a certain point $x = x_0$:

$$\int\limits_a^{*b} f(x)\,dx = \lim_{\epsilon\to 0}\left\{\int\limits_a^{x_0-\epsilon} + \int\limits_{x_0+\epsilon}^{b}\right\} f(x)\,dx.$$

‡ E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 2nd ed. 1948) Chap. V.

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seems not to have been previously observed. Besides, among other things, I show how this theorem and Parseval's formula

$$\int_{-1}^{1} \{\phi_1 \mathscr{T}[\phi_2]+\phi_2 \mathscr{T}[\phi_1]\}\, dx = 0 \tag{6}$$

can be also interpreted as formulae for the change of the order of successive principal integrations, and indicate an asymptotic property of the $\mathscr{T}$-transforms as $x \to \pm 1$.

2. I establish the *convolution theorem* for the $\mathscr{H}$-transforms because in this case it is sufficient to suppose that both functions ϕ_1, ϕ_2 vanish identically outside $(-1, 1)$ to obtain the corresponding result for the $\mathscr{T}$-transforms.

THEOREM. *Let the functions $\phi_1(x)$ and $\phi_2(x)$ belong in $(-\infty, \infty)$ to the classes L^{p_1} and L^{p_2} respectively ($p_1 > 1$, $p_2 > 1$). Then, if*

$$\frac{1}{p_1}+\frac{1}{p_2} < 1, \tag{7}$$

we have almost everywhere

$$\mathscr{H}\{\phi_1 \mathscr{H}[\phi_2]+\phi_2 \mathscr{H}[\phi_1]\} = \mathscr{H}[\phi_1]\mathscr{H}[\phi_2]-\phi_1 \phi_2. \tag{8}$$

Let $f_1(x)$, $f_2(x)$ be the $\mathscr{H}$-transforms of $\phi_1(x)$, $\phi_2(x)$ respectively and consider the two analytic functions

$$\Phi_1(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{f_1(t)+i\phi_1(t)}{t-z}\, dt, \qquad \Phi_2(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{f_2(t)+i\phi_2(t)}{t-z}\, dt$$

which are regular in the half-plane $y = \operatorname{im} z > 0$ and satisfy† there (uniformly) the conditions

$$\int_{-\infty}^{\infty} |\Phi_1(x+iy)|^{p_1}\, dx < K_1, \qquad \int_{-\infty}^{\infty} |\Phi_2(x+iy)|^{p_2}\, dx < K_2$$

where K_1 and K_2 are two positive constants, since $f_1(x)$ and $f_2(x)$ belong also to the classes L^{p_1}, L^{p_2} respectively.‡

Consequently, if, considering (7), we put

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{q} \quad (q > 1),$$

the analytic function $\qquad \Phi(z) = \Phi_1(z)\Phi_2(z)$

† Titchmarsh, op. cit. 139 (Th. 103 can be inverted).
‡ Ibid. 132.

should satisfy the similar condition

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^q \, dx < K_1^{q/p_1} K_2^{q/p_2}$$

because Hölder's inequality allows us to write

$$\int_{-\infty}^{\infty} |\Phi_1(x+iy)|^q |\Phi_2(x+iy)|^q \, dx$$
$$\leqslant \Big[\int_{-\infty}^{\infty} |\Phi_1(x+iy)|^{pq} \, dx \Big]^{1/p} \Big[\int_{-\infty}^{\infty} [\Phi_2(x+iy)|^{p'q} \, dx \Big]^{1/p}.$$

where
$$p = \frac{p_1}{q}, \qquad p' = \frac{p}{p-1} = \frac{p_2}{q}.$$

Hence we can apply to the function $\Phi(z)$ the basic theorem on the boundary values of an analytic function regular in a half-plane† and we see thus that *almost everywhere*

$$\mathrm{re}\,\Phi(x+i0) = \mathscr{H}_x[\mathrm{im}\,\Phi(y+i0)]$$

that is

$$f_1(x)f_2(x) - \phi_1(x)\phi_2(x) = \mathscr{H}_x[f_1(y)\phi_2(y) + f_2(y)\phi_1(y)],$$

which is nothing else than (8).

In particular, if $\phi_1(x)$ and $\phi_2(x)$ vanish identically outside of $(-1, 1)$ we obtain (5).

3. The importance of (8) arises from its equivalence with the most important particular case

$$\int_a^{*b} \frac{\phi_1(z)}{z-x} \, dz \int_a^{*b} \frac{\phi_2(y)}{y-z} \, dy$$
$$= \int_a^{*b} \phi_2(y) \, dy \int_a^{*b} \frac{\phi_1(z) \, dz}{(z-x)(y-z)} - \pi^2 \phi_1(x)\phi_2(x) \quad (a < x < b) \tag{9}$$

of the formula for the change of the order of two successive principal integrations

$$\int_a^{*b} \frac{dz}{z-x} \int_a^{*b} \frac{F(x,y,z)}{y-z} \, dy = \int_a^{*b} dy \int_a^{*b} \frac{F(x,y,z)}{(z-x)(y-z)} \, dz - \pi^2 F(x,x,x), \tag{10}$$

a formula which is usually credited to Poincaré (1910),‡ but is to be

† Titchmarsh, op. cit. Th. 103, p. 139.

‡ *Leçons de Mécanique Céleste*, t. III, 253. (The formula there contains an error of sign.)

found already in a paper of 1908 of G. H. Hardy† and still lacks (if I am not wrong) a modern proof free from unnecessary restrictions on the function $F(x,y,z)$.‡

In fact, considering that

$$\frac{1}{(z-x)(y-z)} = \frac{1}{y-x}\left(\frac{1}{z-x}-\frac{1}{z-y}\right),$$

the formula (9) with $a = -\infty$, $b = \infty$—for otherwise we may suppose $\phi_1 \equiv \phi_2 \equiv 0$ outside of (a,b)—can be also written

$$\pi^2\mathscr{H}_x\{\phi_1(z)\mathscr{H}_z[\phi_2(y)]\}$$
$$= \pi^2\mathscr{H}_x[\phi_2(y)]\mathscr{H}_x[\phi_1(z)]-\pi^2\mathscr{H}_x\{\phi_2(y)\mathscr{H}_y[\phi_1(z)]\}-\pi^2\phi_1(x)\phi_2(x),$$

and in this form it coincides with (8) multiplied by π^2. Hence *the inversion formula* (9) *can be considered as true almost everywhere, provided that the functions* $\phi_1(x)$, $\phi_2(x)$ *belong to the classes* L^{p_1}, L^{p_2} *respectively, with* $1/p_1+1/p_2 < 1$.

In a similar manner Parseval's formula (6) is equivalent to the inversion formula

$$\int\limits_a^b \phi_1(x)\,dx \int\limits_a^{*b} \frac{\phi_2(y)}{y-x}\,dy = \int\limits_a^b \phi_2(y)\,dy \int\limits_a^{*b} \frac{\phi_1(x)}{y-x}\,dx, \tag{11}$$

which is hence assured so long as the functions $\phi_1(x)$, $\phi_2(x)$ *belong to the classes* L^{p_1}, L^{p_2} *respectively, with* $1/p_1+1/p_2 \leqslant 1$.§

4. Now, in order to resolve the *airfoil equation* (4)‖ in the space L^p $(p > 1)$, we observe firstly that, with the help of the substitution

† 'The theory of Cauchy's principal values (fourth paper)', *Proc. London Math. Soc.* (2) 7, 181–208.

‡ A proof of (10) that is rigorous but not simple and imposes strong conditions on the function F is contained in my paper of 1923 on partial differential equations of *mixed type*. [*Mem. Acc. Lincei Roma* (5) 14 (1923), 133–267—Russ. transl. 1947—Engl. transl. 1948, Brown University (U.S.A.)]. The proof given by Hardy (op. cit.) is of the same kind.

§ We see thus that, under certain conditions, the order of a principal integration and an ordinary integration can be freely interchanged. From a general point of view this question coincides with that of the extension of the classic Fubini's theorem to the functions with a *line* of points of infinity (of the first order).

‖ This equation plays an important role in Aerodynamics. See e.g. E. Reissner, 'Boundary value problems in aerodynamics of lifting surfaces in non-uniform motion', *Bull. American Math. Soc.* 55 (1949), 825–50. In 1939 H. Söhngen [*Math. Zeits.* 45 (1939), 245–64] gave an elegant method of solution founded on the use of Poisson's integral of the potential theory. This method leads naturally to the same formulae given here but under essentially stronger conditions.

$y = (1-t^2)/(1+t^2)$ we have

$$\mathscr{T}_x[(1-y^2)^{-\frac{1}{2}}] = \frac{2}{\pi} \int_0^{*\infty} \frac{dt}{(1-x)-(1+x)t^2} = 0 \quad (-1 < x < 1), \tag{12}$$

and consequently also

$$\mathscr{T}_x[(1-y^2)^{\frac{1}{2}}] = \frac{1}{\pi} \int_{-1}^{*1} \frac{(1-x^2)-(y^2-x^2)}{\sqrt{(1-y^2)}(y-x)}\, dy = -\frac{1}{\pi} \int_{-1}^{1} \frac{x+y}{\sqrt{(1-y^2)}}\, dy = -x. \tag{13}$$

Successively we apply the convolution theorem (5) to the pair of functions

$$\phi_1(x) = \sqrt{(1-x^2)}, \qquad \phi_2(x) = \phi(x),$$

which is certainly correct because the function $\phi_1(x)$, being bounded, belongs to the class $L^{p'}$ with any large p'. We have thus (almost everywhere)

$$\mathscr{T}_x[-y\phi(y)+\sqrt{(1-y^2)}f(y)] = -xf(x)-\sqrt{(1-x^2)}\phi(x).$$

But, on the other hand,

$$\mathscr{T}_x[y\,\phi(y)] = \frac{1}{\pi} \int_{-1}^{*1} \frac{y-x+x}{y-x} \phi(y)\, dy = x\mathscr{T}_x[\phi(y)] + \frac{1}{\pi} \int_{-1}^{1} \phi(y)\, dy.$$

Hence

$$\begin{aligned}\sqrt{(1-x^2)}\phi(x) &= x\{\mathscr{T}_x[\phi(y)]-f(x)\} - \mathscr{T}_x[\sqrt{(1-y^2)}f(y)] + \frac{1}{\pi} \int_{-1}^{1} \phi(y)\, dy \\ &= -\mathscr{T}_x[\sqrt{(1-y^2)}f(y)] + \frac{1}{\pi} \int_{-1}^{1} \phi(y)\, dy,\end{aligned}$$

and we obtain

$$\begin{aligned}\phi(x) &= -\frac{1}{\sqrt{(1-x^2)}} \mathscr{T}_x[\sqrt{(1-y^2)}f(y)] + \frac{C}{\sqrt{(1-x^2)}} \\ &= -\frac{1}{\pi} \int_{-1}^{*1} \frac{\sqrt{(1-y^2)}f(y)}{\sqrt{(1-x^2)}(y-x)}\, dy + \frac{C}{\sqrt{(1-x^2)}},\end{aligned} \tag{14}$$

where, considering (12), the constant

$$C = \frac{1}{\pi} \int_{-1}^{1} \phi(y)\, dy \tag{15}$$

assumes the character of an arbitrary constant.

5. The precise significance of the previous result is naturally the following: *If the given equation has a solution of the class L^p ($p > 1$), then this solution must necessarily have the form* (14).

Consequently we can consider already as granted that *the homogeneous equation*

$$\frac{1}{\pi}\int\limits_{-1}^{*1}\frac{\phi(y)}{y-x}\,dy = 0 \quad (-1 > x > 1) \tag{16}$$

has the non-trivial solutions $C(1-x^2)^{-\frac{1}{2}}$ *of the class* L^p $(p > 1)$ *only*; but it is not yet proved that the first term on the right of (15) satisfies as a matter of fact the given equation (4).

In order to prove this later, we must show firstly that the said term, i.e. *the function*

$$\phi_0(x) = \frac{1}{\pi}\int\limits_{-1}^{1}\frac{\sqrt{(1-x^2)}-\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}(y-x)}f(y)\,dy - \frac{1}{\pi}\int\limits_{-1}^{*1}\frac{f(y)}{y-x}\,dy$$

belongs to the class L^q *with* $1 < q < \frac{4}{3}$, *provided that the given function* $f(x)$ *belongs to the class* L^p *with* $p > \frac{4}{3}$.

For this, considering that

$$\phi_0(x) = \frac{1}{\pi}(1-x^2)^{-\frac{1}{2}}\int\limits_{-1}^{1}\frac{(x+y)f(y)}{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}\,dy - \mathscr{T}_x[f(y)],$$

the sole difficulty is the determination of the class of the function

$$g(x) = \int\limits_{-1}^{1}\frac{(x+y)f(y)}{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}\,dy$$

because the class of the factor $(1-x^2)^{-\frac{1}{2}}$ is obviously $2-\epsilon$ $(\epsilon > 0)$, while the class of $\mathscr{T}[f]$ (as is likely also in the case of $\mathscr{H}[f]$) is the same as that of f.† But, using Hölder's inequality, we have

$$|g(x)|^{p'} \leqslant \int\limits_{-1}^{1}\left[\frac{|x+y|}{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}\right]^{p'}dy\left[\int\limits_{-1}^{1}|f(y)|^p\,dy\right]^{p'/p} \quad \left(p' = \frac{p}{p-1}\right),$$

and consequently

$$\int\limits_{-1}^{1}|g(x)|^{p'}\,dx \leqslant \int\limits_{-1}^{1}\int\limits_{-1}^{1}\left[\frac{|x+y|}{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}\right]^{p'}dxdy\left[\int\limits_{-1}^{1}|f(y)|^p\,dy\right]^{p'/p}.$$

Hence the function $g(x)$ belongs to the class $L^{p'}$ so long as the double integral on the right side is finite, i.e. so long as $p' < 4$, i.e. provided that $p > \frac{4}{3}$, since the integrand becomes infinite (of the order $\frac{1}{2}p'$) only at the two boundary points $x = y = \pm 1$ of the square

$$(-1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1).$$

† Titchmarsh, op. cit. 132.

Remembering the rule for the class of the product of two functions we thus see that the product $(1-x^2)^{-\frac{1}{4}}g(x)$ and consequently the function $\phi_0(x)$ belong as a matter of fact to the class L^q with $1 < q < \frac{4}{3}$ provided that $p > \frac{4}{3}$.

6. Now we can apply the convolution theorem to the pair

$$\phi_1(x) = \sqrt{(1-x^2)}, \qquad \phi_2(x) = \phi_0(x),$$

and in a manner similar to that used above to obtain (14) we obtain

$$\mathcal{T}_x\{\sqrt{(1-y^2)}\mathcal{T}_y[\phi_0(z)]\}$$

$$= \frac{1}{\pi}\int\limits_{-1}^{1} \phi_0(y)\,dy - \sqrt{(1-x^2)}\phi_0(x) = C_0 - \sqrt{(1-x^2)}\phi_0(x).$$

But $\phi_0(x)$ has the expression (14). Hence

$$\mathcal{T}_x\{\sqrt{(1-y^2)}\mathcal{T}_y[\phi_0(z)]\} = C_0 + \mathcal{T}_x[\sqrt{(1-y^2)}f(y)] - C:$$

that is $$\mathcal{T}_x[\sqrt{(1-y^2)}\{\mathcal{T}_y[\phi_0(z)] - f(y)\}] = 0,$$

provided that the arbitrary constant C has just the previous value C_0.†

Using the previous result about the homogeneous equation (16) we see thus that we must have necessarily

$$\sqrt{(1-x^2)}\{\mathcal{T}_x[\phi_0(z)] - f(x)\} = K/\sqrt{(1-x^2)}:$$

that is $$\mathcal{T}_x[\phi_0(y)] - f(x) = K/(1-x^2), \tag{17}$$

where K is a suitable constant. But as long as $K \neq 0$ the function on the right of (17) is not summable in $(-1, 1)$ while the function on the left belongs to the class L^q with $1 < q < \frac{4}{3}$. Hence $K = 0$ and this proves that, as a matter of fact, we have almost everywhere

$$\mathcal{T}_x[\phi_0(y)] \equiv f(x).$$

We have thus proved that

As long as the given function $f(x)$ belongs to the class L^p with $p > \frac{4}{3}$, the airfoil equation (14) *admits the solution*

$$\phi(x) = -\frac{1}{\pi}\int\limits_{-1}^{*1} \frac{\sqrt{(1-y^2)}f(y)}{\sqrt{(1-x^2)}(y-x)}\,dy + \frac{C}{\sqrt{(1-x^2)}}, \tag{18}$$

where C is an arbitrary constant, and the first term belongs to the class $L^{\frac{4}{3}-\epsilon}$ $(\epsilon > 0)$ at least.

Moreover, the second term—which belongs to the class $L^{2-\epsilon}$ $(\epsilon > 0)$—represents the unique solution of the corresponding homogeneous equation in the space L^p $(p > 1)$.

† This is obviously permissible because the term with the constant C is without importance by the verification of the formula (14) since $\mathcal{T}[(1-y^2)^{-\frac{1}{2}}] \equiv 0$.

If $f(x)$ belongs to a class L^p with $1 < p \leqslant \frac{4}{3}$, we can no longer affirm (in the previous way) that the function $\phi_0(x)$ belongs certainly to the class $L^{1+\epsilon}$, but it remains true that (i) if the given equation has a solution of the class $L^{1+\epsilon}$, this must necessarily have the form (18); (ii) if the function $\phi_0(x)$ belongs to the class $L^{1+\epsilon}$, it necessarily satisfies the given equation.

7. It is interesting to notice that so long as the function $\phi(x)$ belongs to the class $L^{2+\epsilon}$ (i.e. to the class L^p with $p > 2$) its $\mathscr{T}$-transform $f(x)$, which belongs to the class $L^{2+\epsilon}$ too, necessarily satisfies the orthogonality condition

$$\int\limits_{-1}^{*1} (1-x^2)^{-\frac{1}{2}} f(x)\,dx = 0. \tag{19}$$

In fact, under the previous hypothesis we can apply Parseval's formula (6) to the pair of functions

$$\phi_1(x) = (1-x^2)^{-\frac{1}{2}}, \qquad \phi_2(x) = \phi(x)$$

and, using (12), we obtain thus (19) without more ado.

Among other things this allows us to put the solution (18) under the alternate form

$$\phi(x) = -\frac{1}{\pi}\int\limits_{-1}^{*1} \frac{\sqrt{(1-x^2)}f(y)}{\sqrt{(1-y^2)}(y-x)}\,dy + \frac{C'}{\sqrt{(1-x^2)}} \tag{20}$$

because, as long as (19) is valid, we have also

$$\int\limits_{-1}^{*1} \left(\sqrt{\left(\frac{1-x^2}{1-y^2}\right)} - \sqrt{\left(\frac{1-y^2}{1-x^2}\right)}\right)\frac{f(y)}{y-x}\,dy$$

$$= \frac{1}{\sqrt{(1-x^2)}}\int\limits_{-1}^{1} \frac{x+y}{\sqrt{(1-y^2)}} f(y)\,dy = \frac{k}{\sqrt{(1-x^2)}}$$

with
$$k = \int\limits_{-1}^{1} \frac{y\,f(y)}{\sqrt{(1-y^2)}}\,dy.$$

Moreover, independently of the condition (19), if at least one of the two functions

$$\frac{f(x)}{\sqrt{(1+x)}}, \quad \frac{f(x)}{\sqrt{(1-x)}}$$

is summable, by virtue of the identities

$$\sqrt{\left(\frac{1-y^2}{1-x^2}\right)} = \sqrt{\left(\frac{1+x}{1-x}\right)}\sqrt{\left(\frac{1-y}{1+y}\right)}\left(1+\frac{y-x}{1+x}\right)$$

$$= \sqrt{\left(\frac{1-x}{1+x}\right)}\sqrt{\left(\frac{1+y}{1-y}\right)}\left(1-\frac{y-x}{1-x}\right),$$

the solution (18) can be put under the two further alternative forms

$$\left.\begin{aligned} \phi(x) &= -\frac{1}{\pi}\sqrt{\left(\frac{1+x}{1-x}\right)}\int\limits_{-1}^{*1}\frac{\sqrt{(1-y)f(y)}}{\sqrt{(1+y)(y-x)}}\,dy+\frac{C''}{\sqrt{(1-x^2)}} \\ \phi(x) &= -\frac{1}{\pi}\sqrt{\left(\frac{1-x}{1+x}\right)}\int\limits_{-1}^{*1}\frac{(1+y)f(y)}{\sqrt{(1-y)(y-x)}}\,dy+\frac{C'''}{\sqrt{(1-x^2)}} \end{aligned}\right\}. \tag{21}$$

8. Some authors† have solved the airfoil equation in another way by use of trigonometrical series, which theoretically is much less satisfying than the present one, but practically may be sometimes useful. This method is based on the use of the formula

$$\int\limits_{0}^{*\pi}\frac{\cos n\eta\,d\eta}{\cos\eta-\cos\xi} = \pi\frac{\sin n\xi}{\sin\xi} \quad (n = 0, 1, 2,\ldots), \tag{22}$$

which, together with the similar formula

$$\int\limits_{0}^{*\pi}\frac{\sin(n+1)\eta\sin\eta}{\cos\eta-\cos\xi}\,d\eta = -\pi\cos(n+1)\xi, \tag{23}$$

shows that the $\mathscr{T}$-transformation operates in a particularly simple manner on Tchebycheff's polynomials

$$T_n(\cos\xi) = \cos n\xi, \qquad U_n(\cos\xi) = \frac{\sin(n+1)\xi}{\sin\xi}.$$

Precisely we have

$$\mathscr{T}_x[(1-y^2)^{-\frac{1}{2}}T_n(y)] = U_{n-1}(x), \qquad \mathscr{T}_x[(1-y^2)^{\frac{1}{2}}U_{n-1}(y)] = -T_n(x)$$
$$(n = 1, 2,\ldots). \tag{24}$$

These formulae are particular cases of more general ones which can be obtained starting from the well-known relation‡ between Jacobi polynomials $P_n^{(\alpha,\beta)}$ and corresponding functions of the second kind $Q_n^{(\alpha,\beta)}$

$$(z-1)^\alpha(z+1)^\beta Q_n^{(\alpha,\beta)}(z) = -\tfrac{1}{2}\int\limits_{-1}^{1}(1-t)^\alpha(1+t)^\beta P_n^{(\alpha,\beta)}(t)\frac{dt}{t-z}$$
$$(\alpha > -1,\ \beta > -1).$$

Here the function on the left is an analytical function $\Phi(z)$ of z, regular

† See e.g. G. Hamel, *Integralgleichungen* (Berlin, 1937), 145.

‡ See e.g. G. Szegő, *Orthogonal Polynomials* (New York, 1939), 73 ff.

in the half-plane $\operatorname{im} z > 0$, which can be expressed by means of hypergeometric functions as follows:

$$\frac{2^{-n-\alpha-\beta}\Gamma(2n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\Phi(z)$$

$$= (z-1)^{-n-1}F\left(n+1, n+\alpha+1; 2n+\alpha+\beta+2; \frac{2}{1-z}\right)$$

$$= (z+1)^{-n-1}F\left(n+1, n+\beta+1; 2n+\alpha+\beta+2; \frac{2}{1+z}\right).$$

Among other things this shows that the function

$$\Phi(x+i0) \equiv u(x)+iv(x)$$

to which $\Phi(z)$ reduces on the real axis is certainly real outside the interval $(-1, 1)$ since

$$0 < \frac{2}{1-x} < 1 \quad (x < -1), \qquad 0 < \frac{2}{1+x} < 1 \quad (x > 1).$$

On the contrary the function $\Phi(x+i0)$ is complex inside $(-1, 1)$, where, using the well-known relation between hypergeometric functions of the arguments z and $1-z$, we have

$$\Phi(x+i0) = 2^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-x}{2}\right)-$$

$$-\frac{\pi}{2\sin\alpha\pi}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)e^{\alpha\pi i},$$

the case of integer α being excluded.† Moreover, the real and imaginary parts $u(x)$ and $v(x)$ of the function $\Phi(x+i0)$ certainly belong to the class L^p $(p > 1)$ because the function $\Phi(z)$ vanishes at infinity like $1/|z|$ and in the neighbourhoods of its singular points $z = \pm 1$ is of the order of $(z-1)^{\alpha}$ and $(z+1)^{\beta}$ respectively.‡

Consequently the function satisfies the condition

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^p \, dx < K$$

and we can use the fundamental formula $u = \mathscr{H}[v]$, which in this case becomes $u = \mathscr{T}[v]$ since $v(x)$ vanishes outside $(-1, 1)$.

† In this case a logarithmic term would be present.
‡ Szegő, op. cit. 77.

We obtain thus the interesting formula

$$\mathscr{T}_x[(1-y)^\alpha(1+y)^\beta P_n^{(\alpha,\beta)}(y)] = \cot\alpha\pi(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)-$$

$$-\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(n+\beta+1)}{\pi\Gamma(n+\alpha+\beta+1)}F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}\right)$$

$$(\alpha>-1,\beta>-1;\ \alpha\neq 0,1,2,...), \tag{25}$$

where the second term is also a polynomial if $\alpha+\beta$ is an *integer* larger than $-n$, for instance in the case $\alpha=\beta=\pm\frac{1}{2}$ (Tchebycheff polynomials).

Another interesting particular case is the case $n=0$, $\beta=-\alpha$, in which

$$P_n^{(\alpha,\beta)}(x)\equiv 1,\qquad F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}\right)\equiv 1$$

and consequently

$$\mathscr{T}_x\left[\left(\frac{1-y}{1+y}\right)^\alpha\right] = \cot\alpha\pi\left(\frac{1-x}{1+x}\right)^\alpha-\frac{1}{\sin\alpha\pi}\quad(0<|\alpha|<1). \tag{26}$$

9. The last formulae are interesting especially because they show that, at least in some cases, a function $\phi(x)$ which becomes infinite like $A(1-x)^{-\alpha}$ or $A(1+x)^{-\alpha}$ $(0<\alpha<1)$ as $x\to\pm1$ is carried by the $\mathscr{T}$-transformation into a function $f(x)$ with similar behaviour, neglecting the fact that A is replaced by $\pm A\cot\alpha\pi$.

This fact is quite general as is shown by the following asymptotic theorem:

THEOREM. *Let the L^p-function $\phi(x)$ $(p>1)$ be representable in an arbitrary small neighbourhood $(-1,-1+\delta)$ of the point $x=-1$ $(\delta>0)$ by a formula of the type*

$$\phi(x) = A(1+x)^{-\alpha}+\psi(x)\quad(0\leqslant\alpha<1), \tag{27}$$

where A denotes any constant and $\psi(x)$ a function vanishing at $x=-1$ and satisfying (uniformly) a Lipschitz condition of positive order ϵ

$$|\psi(x)-\psi(x_0)|<K|x-x_0|^\epsilon. \tag{28}$$

Then the $\mathscr{T}$-transform $f(x)$ of $\phi(x)$ has the asymptotic representation

$$f(x) = A\cot\alpha\pi(1+x)^{-\alpha}+O(1)\quad(x\to-1) \tag{29}$$

if $0<\alpha<1$, and

$$f(x) = -\frac{A}{\pi}\log(1+x)+O(1)\quad(x\to-1) \tag{30}$$

if $\alpha=0$.

If the point $x=-1$ is replaced by the point $x=1$ all remains the same except that $\cot\alpha\pi$ is changed into $-\cot\alpha\pi$ and $-\log(1+x)$ into $+\log(1-x)$.

For the proof we observe firstly that, as long as $f(x)$ becomes infinite at $x = 1$, its asymptotic behaviour for $x \to -1$ depends exclusively on the values of $\phi(x)$ in the arbitrary small interval $(-1, -1+\delta)$ $(\delta > 0)$ because, if—supposing x outside $(-1, 1)$—we put

$$f(x) = \mathscr{T}_x[\phi(y)] = \frac{1}{\pi} \int_{-1}^{*\delta-1} \frac{\phi(y)}{y-x}\,dy + \frac{1}{\pi} \int_{\delta-1}^{1} \frac{\phi(y)}{y-x}\,dy,$$

the second term represents an analytic function regular in the whole x-plane cut along the segment $(-1+\delta, 1)$ of the real axis; hence regular even in the neighbourhood of $x = -1$.

Thereafter it is not substantially restrictive to suppose that (27) is valid in the whole interval $(-1, 1)$ or, better still, to suppose that we have there

$$\phi(x) = \frac{A}{2^\alpha}\left(\frac{1-x}{1+x}\right)^\alpha + \psi^*(x),$$

where the function

$$\psi^*(x) = \psi(x) + \frac{A}{(1+x)^\alpha}\left[1 - \left(\frac{1-x}{2}\right)^\alpha\right]$$

satisfies a similar condition to the previous $\psi(x)$, with inclusion of the vanishing at $x = -1$. But, by (26), as long as $0 < \alpha < 1$ we have

$$\mathscr{T}_x\left[\frac{A}{2^\alpha}\left(\frac{1-y}{1+y}\right)^\alpha\right]$$

$$= \frac{A}{2^\alpha}\cot\alpha\pi\left(\frac{1-x}{1+x}\right)^\alpha - \frac{1}{\sin\alpha\pi} = A\cot(\alpha\pi)(1+x)^{-\alpha} + O(1)$$

$$(x \to -1),$$

while for $\alpha = 0$ we have

$$\pi\mathscr{T}_x[A] = A\log\frac{1-x}{1+x} = -A\log(1+x) + O(1) \quad (x \to -1).$$

Hence it only remains to show that

$$\mathscr{T}_x[\psi^*(y)] = O(1) \quad (x \to -1),$$

which is an immediate consequence of (28) since

$$\pi\mathscr{T}_x[\psi^*(y)]$$

$$= \int_{-1}^{*1} \frac{\psi^*(y) - \psi^*(x) + \psi^*(x)}{y-x}\,dy = \psi^*(x)\log\frac{1-x}{1+x} + \int_{-1}^{1} \frac{\psi^*(y) - \psi^*(x)}{y-x}\,dy.$$

In fact we have

$$\left|\int_{-1}^{1} \frac{\psi^*(y)-\psi^*(x)}{y-x}\,dy\right| < K\int_{-1}^{1} |y-x|^{\epsilon-1}\,dy < \frac{K}{\epsilon}2^{\epsilon+1}$$

and, on the other hand,

$$\left|\psi^*(x)\log\frac{1-x}{1+x}\right| = |\psi^*(x)-\psi^*(-1)|\left|\log\frac{1-x}{1+x}\right|$$
$$< K(1+x)^\epsilon[\log 2-\log(1+x)].$$

The passage from the case $x \to -1$ to the case $x \to 1$ offers no difficulties, but, by the formula (26), we must change α into $-\alpha$, and this explains the change of $\cot\alpha\pi$ into $-\cot\alpha\pi$.

A PROPERTY OF PROJECTED SEGRE VARIETIES

By J. G. SEMPLE (*London*)

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1. Let ϕ be a projected Veronese surface in S_4, model of the conics outpolar to a conic e in a plane π. It is well known* then that the trisecant lines of ϕ—which are mapped in π by triads of points self-conjugate for e—form a general linear line-congruence of S_4; also that there are among them ∞^1 inflexional lines which touch a unique inflexional ${}^0C^4$, image of e on ϕ; and lastly that ϕ is uniquely determined by this inflexional curve, being the locus of intersections of pairs of its osculating planes.

In this note I shall show that these results have close analogues when ϕ is replaced by the general projection on S_7 of a Segre variety V_4^6 of S_8; and I also consider, more generally, some properties of analogous projected Segre varieties in S_{r^2+2r-1}.

2. The projected Segre variety $\Omega(r)$

Let $V(r)$ be the Segre variety—of dimension $2r$ in space S_{r^2+2r}—which maps, for example, the pairs of points of spaces S_r and S'_r; and let $\Omega(r)$ be its projection (obviously non-singular) from a general point O of the surrounding space on to a space S_{r^2+2r-1}. We note first, then, that

The $(r+1)$*-secant* $[r-1]$*'s of* $\Omega(r)$ *generate a congruence of order* 1, *i.e. precisely one of them passes through a general point of* S_{r^2+2r-1}.

This is an immediate consequence of the fact that $V(r)$ has a unique $(r+1)$-secant $[r]$ passing through a general line of the surrounding space† S_{r^2+2r}.

* For these properties of the projected Veronese surface see, for example, Telling, *The rational quartic curve* (Cambridge, 1936), or Semple and Roth, *Algebraic Geometry* (Oxford, 1949), pp. 153 and 276. If the point K and conic k correspond on ϕ to pole and polar for e, then K is the residual intersection of ϕ with the plane of k; if K lies on k, then its locus is the inflexional curve E corresponding to e, and the plane of k touches ϕ and osculates E at K.

† If non-null square matrices (a_{ij}) of order $r+1$ are mapped by the points of S_{r^2+2r}, with homogeneous coodinates a_{ij}, then matrices of rank 1 are mapped by the points of a Segre variety $V(r)$. The property stated in the text corresponds then to the fact that all the matrices of a *general* pencil are linearly dependent on $r+1$ fixed matrices of unit rank. For the case $r = 2$, see also Semple, 'Properties of certain cubic primals', *Quart. J. of Math.* (Oxford), 15 (1944), 26–33; 29.

We now regard $V(r)$, as we may, as the model of all point-prime *couples* (P, Π) of one space S_r, its parametric equations being

$$X_{ij} = x_i u_j \quad (i, j = 0,..., r),$$

where x_i and u_j are the coordinates of P and Π respectively. The point O given by

$$X_{00} = X_{11} = ... = X_{rr}, \qquad X_{ij} = 0 \quad (i \neq j)$$

is generally situated with respect to $V(r)$; and we may therefore take $\Omega(r)$ to be the projection of $V(r)$ from O on to the prime S_{r^2+2r-1} whose equation is

$$X_{00}+X_{11}+...+X_{rr} = 0.$$

This projection $\Omega(r)$ is also an unexceptional model of the couples (P, Π) of S_r.

If E is the section of $V(r)$ by S_{r^2+2r-1}, then, plainly, E lies on $\Omega(r)$, and its points map the *incident couples* $(\overline{P, \Pi})$ of S_r, for which P lies on Π.

If K is the section of $V(r)$ by the prime whose coordinate matrix is $\mathbf{U}$, the points of K map the couples which are *conjugate with respect to the fixed collineation* ϖ of S_r with matrix $\mathbf{U}'$; that is to say, the couples are all those for which Π passes through the transform of P in ϖ. Further, the prime passes through O if and only if $\mathbf{U}'$ has zero trace. Hence

Every prime section of $\Omega(r)$ *maps the system of couples* (P, Π) *which are conjugate with respect to a collineation of* S_r *whose matrix is of zero trace; and conversely.*

Consider now the double condition on two couples (P, Π) and (P', Π') which requires P to lie on Π' and P' to lie on Π. If two couples satisfy this condition, I shall say that they are *related*, and I shall say also that the points which represent them on $\Omega(r)$ are *related points of* $\Omega(r)$.

The points of $\Omega(r)$ which are related to a fixed point W generate a manifold w—of dimension two less than that of $\Omega(r)$—which is evidently a Segre variety $V(r-1)$; and, conversely, every Segre $V(r-1)$ of $\Omega(r)$ arises in this way. We call w the *polar variety* of W on $\Omega(r)$, noting that the correspondence between W and w is $(1, 1)$, so that every w has a unique *pole* W. The principal property of the representation is as follows:

The pole W *of any one of the Segre varieties* w *is the unique residual point in which the* $[r^2-1]$ *containing* w *meets* $\Omega(r)$.

This follows from the fact, which is easy to verify, that if ϖ is any collineation of S_r with respect to which all couples related to a fixed couple (P, Π) are conjugate, then the matrix of ϖ is at most of rank 2,

and it has zero trace if and only if the fixed couple (P, Π) is itself conjugate with respect to ϖ.

Suppose now that $W_0 W_1 ... W_r$ is any general $(r+1)$-secant $[r-1]$ of $\Omega(r)$. Through $W_1,..., W_r$, as in general through any r points of $\Omega(r)$, there passes a unique Segre variety, w_0 say. This must have W_0 as its pole, since W_0 is linearly dependent on $W_1,..., W_r$ but does not lie on w_0. This implies generally, then, that all the points W_i are related to each other on $\Omega(r)$. Hence

The $(r+1)$-secant $[r-1]$'s of $\Omega(r)$ map the simplexes of S_r, in the sense that the $r+1$ couples (P_i, Π_i) formed by the vertices and opposite faces of a simplex represent the intersections of $\Omega(r)$ with an $(r+1)$-secant $[r-1]$.

We note also that the $(r+1)$-secant $[r-1]$'s through any point W of $\Omega(r)$ are the r-secant $[r-1]$'s of the polar variety w which pass through W; these all lie in an $[r^2-1]$ and they project from W into the r-secant $[r-2]$'s of a manifold $\Omega(r-1)$.

The *coincidence locus* on $\Omega(r)$, locus of self-related points W, is clearly the manifold E which maps the united couples $(\overline{P, \Pi})$ of S_r. If W is any point of E, its polar variety w passes through it and touches E; and, in fact, the two generating $[r-1]$'s of E at W are the same as those of w through the same point.

3. The congruence of trisecants of a $V_4^6[7]$

The simplest example of the results we have obtained concerns the manifold $\Omega(2)$ which is the general projection on S_7 of the Segre V_4^6 of S_8.

The varieties w on $\Omega(2)$ are ∞^4 quadric surfaces, of which one passes through two general points of $\Omega(2)$; and the pole W of any one of them is the residual point in which its containing solid meets $\Omega(2)$.

The congruence (of order 1) of ∞^6 trisecant lines of $\Omega(2)$ is such that every point W of $\Omega(2)$ is vertex of a solid star $W(\Sigma)$ of trisecants, where Σ is the solid which contains W and its polar surface w.

The points of $\Omega(2)$, as we have seen, can be mapped on the point-line couples (P, p) of a plane π in such a way that to a point W and its polar surface w there correspond respectively a couple (P, p) and the class of couples related to (P, p); and in this representation the trisecants of $\Omega(2)$ are mapped by triangles of π, each vertex of a triangle being coupled with the opposite side.

The polarity (W, w) on $\Omega(2)$ has a coincidence locus E which is the $V_3^6[7]$ representing first-order line-elements $(\overline{P, p})$ of π. E has, at each of its points, a *focal plane*—the plane of the two generating lines through

the point; and Severi* has shown that a curve on E represents the set of line elements of a curve in π if and only if it is a *focal curve* of E, i.e. touches the focal plane of E at each point of itself.

If W lies on E, its polar surface w passes through it and touches the focal plane of E there; also, as is easy to prove, the solid Σ containing W and w lies in the tangent [4] to $\Omega(2)$ at W, being distinct, however, from the tangent solid of E at the same point. Hence

The ∞^5 *tangential trisecants of* $\Omega(2)$ *are all the trisecants which issue from points of* E*; they touch* $\Omega(2)$ *at these points and generate a solid-star at each. The* ∞^4 *inflexional trisecants are all the focal tangent lines of* E*, i.e. the tangent lines which lie in the focal planes at their points of contact.*

Plainly, if w is the polar surface of W, it meets E in the conic of contact of tangents from W; and, conversely, any conic of E is the trace on E of a unique surface w.

We may note also that $\Omega(2)$ contains ∞^5 *projected* Veronese surfaces ϕ, each of which represents the system of couples (P, p) in which p is the polar of P for a fixed conic k in π. Each ϕ meets E in its inflexional ${}^0C^4$, this being the focal curve on E which images the system of line-elements of k. Plainly any two surfaces of ϕ meet, in general, in the three intersections of $\Omega(2)$ with a trisecant, and any trisecant of $\Omega(2)$ is a common trisecant of ∞^2 surfaces ϕ.

Since each ϕ is uniquely determined by its inflexional curve, we have the result:

$\Omega(2)$ *is uniquely determined by its coincidence locus* E*, being generated* (*multiply*) *by the* ∞^5 *projected Veronese surfaces* ϕ *whose inflexional curves are the* ∞^5 *focal* ${}^0C^4$ *of* E *which represent conics* (*as systems of line-elements*) *in the plane.*

* 'I fondamenti della geometria numerativa', *Ann. di mat.* (4) 19 (1940), § 61.

ON THE INFLEXIONAL CURVE OF AN ALGEBRAIC SURFACE IN S_4

By B. SEGRE (*Bologna*)

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1. Let F be any algebraic surface belonging to S_4 and P any simple point of F. The pairs of tangents at P to the sections of F by the ∞^1 tangent primes there form an involution I, whose double lines may be called the *principal tangents* of F at P.

The latter are in general distinct, in which case they are two conjugate tangents in the sense of differential geometry; in any case, they are the tangents at P to those prime sections of F which possess a cusp at P. The curves of F whose tangents are principal tangents of F are in general transcendental; they form a *double conjugate system* (or *net*, in the sense of differential geometry), and are precisely the characteristic curves of the partial differential equation of Laplace having F for integral surface. This means that the five homogeneous coordinates x of the general point of F, in a suitably restricted region of the surface, are functions of two parameters u, v, satisfying an equation of the form

$$a(u,v)\frac{\partial^2 x}{\partial u^2}+b(u,v)\frac{\partial^2 x}{\partial u \partial v}+c(u,v)\frac{\partial^2 x}{\partial v^2}+d(u,v)\frac{\partial x}{\partial u}+e(u,v)\frac{\partial x}{\partial v}+f(u,v)x=0.$$

In particular cases this equation may be parabolic, i.e. at every point of F the two principal tangents coincide; we shall then say that F is *parabolic*. This occurs, for example, if F is a ruled surface, the conjugate system being then formed by the generators counted twice; but there may be other cases, which it would be interesting to study under the assumption that F is algebraic.

In the sequel I shall assume that F is *non-parabolic*; we can then consider, on F, the *locus* Γ *of points at which the two principal tangents coincide*. At the general point P of Γ, the involution I is parabolic, i.e. all its pairs contain a fixed tangent, which is the unique principal tangent to F at P; conversely, if I is parabolic, then P lies on Γ. It is immediately seen (e.g. by projecting F upon an S_3 and recalling some elementary properties of differential geometry) that the unique principal tangent at P has 3-point intersection with F at P, i.e. the general prime containing it cuts F in a curve having an inflexion at P. The reciprocal result is also true; Γ is therefore the locus of points of contact of the

inflexional (or asymptotic) tangents of F, and so can be called the *inflexional* (or *asymptotic*) *curve* of F.

It can be shown that

The inflexional tangent of F at a point P of Γ touches Γ at P if and only if there is a prime section of F having a tacnode at P. In general, this condition is not satisfied at every point of Γ.

2. Let us now suppose, in particular, that F is a Segre surface, i.e. the intersection of two quadrics

$$f(x) = 0, \qquad \phi(x) = 0 \tag{1}$$

of S_4. Then it is immediately seen that the inflexional curve of F is formed by the lines of F. It is well known that the number of these lines is 16; we can obtain this result in a more significant form, by proving

THEOREM I. *The inflexional curve of a Segre surface* (1) *is the complete intersection of the surface with a quartic primal g, whose equation has coefficients which are forms of the sixth degree in the coefficients of each of the original forms f, ϕ.*

To prove this, we remark that the join of two *distinct* points x, y lies on F if and only if these two points satisfy the equations (1) and

$$f(y) = 0, \qquad \phi(y) = 0, \qquad f(x,y) = 0, \qquad \phi(x,y) = 0, \tag{2}$$

where $f(x,y)$ and $\phi(x,y)$ are the bilinear polar forms of f and ϕ. Consider now any linear equation connecting the y's

$$\psi(y) = 0, \tag{3}$$

with indeterminate coefficients. On eliminating the y's from the equations (2) and (3), we obtain an equation

$$h(x) = 0, \tag{4}$$

of degree 8 in the x's, whose coefficients are forms of degrees 6, 6, 4 respectively in the coefficients of the forms f, ϕ, ψ.

We now remark that, on identifying the y's with the x's, the equations (2) coincide with the equations (1), and (3) becomes $\psi(x) = 0$. It follows that the form $h(x)$, $\mathrm{mod}(f(x), \phi(x))$, is divisible by $[\psi(x)]^4$; and that, on suppressing this extraneous factor, (4) reduces to the primal g having the characters stated above.*

* The equation of the primal g can be obtained from above, by reducing f and ϕ to sums of squares. I shall, moreover, show in the Addendum how g can be expressed as a covariant of f and ϕ.

The fact that *the* 16 *lines of a Segre surface F are the complete intersection of F with a quartic primal* can also be deduced from the remark that the intersection of F with a general quadric is a curve A of order 8 and genus 5, and that the locus of the ∞^2 chords of A is a primal, of order 16, containing A with multiplicity 6 and intersecting F residually in its 16 lines.

3. A first extension of Theorem I is given by

THEOREM II. *If a surface F of S_4 is the complete intersection of two general primals of orders m ($\geqslant 2$) and n ($\geqslant 2$), then the inflexional curve of F is the complete intersection of F with a primal g of order* $6m+6n-20$.

Let f, ϕ be primals of orders m, n intersecting in F. If P is any point of S_4, I denote by f_P, ϕ_P its polar quadrics with respect to f, ϕ, by F_P the Segre surface intersection of f_P and ϕ_P, and by g_P the quartic primal (obtained in § 2) which intersects F_P along its 16 lines.

In particular, if P is any point of the inflexional curve Γ of F, let l be the inflexional tangent of F at P. Then l lies on f_P and ϕ_P, i.e. on F_P, and therefore also on g_P.

Conversely, if P is a simple point of F (and therefore also of F_P) which lies on g_P, this implies (§ 2) that P is a point of one of the 16 lines of F_P, l say. Hence l lies on both f_P and ϕ_P; this implies, as is easily seen, that l has a 3-point contact at P with both f and ϕ, therefore also with F, so that P lies on Γ.

It follows that Γ is the intersection of F with the locus of the points P of S_4 which lie on the corresponding primal g_P. Now the coefficients of f_P, ϕ_P are forms of degrees $m-2$, $n-2$ in the coordinates of P; hence (§ 2) the quartic primal g_P contains these coordinates to the degree $6(m-2)+6(n-2)$, and so the locus just considered is a primal g of order

$$6(m-2)+6(n-2)+4 = 6m+6n-20.$$

4. As a further extension of Theorem I, I shall now prove

THEOREM III. *Consider in S_4 any non-parabolic irreducible surface F, free from multiple curves, and denote by C, K, Γ respectively a prime section, a (virtual) canonical curve, and the inflexional curve of F. Then these three curves are connected by the linear equivalence*

$$\Gamma \equiv 10C+6K \quad (\textit{on } F). \tag{5}$$

I consider any two primals f, ϕ, of sufficiently high orders m, n, containing F simply, and I denote by D the common curve of F and the residual intersection of f, ϕ. Then, as is well known,

$$D \equiv (m+n-5)C-K.$$

From § 3 we see that the inflexional curve Γ of F can be cut on F by a primal g, of order $6m+6n-20$; but now g contains D, and Γ is its intersection with F residual to the curve D counted a certain number k times. Hence

$$\Gamma \equiv (6m+6n-20)C-kD \equiv [(6-k)(m+n)+(5k-20)]C+kK.$$

The coefficient of C on the right-hand side cannot depend on m, n, and so $k = 6$, whence the equivalence (5).*

Intersecting (5) by C, and equating the numbers of points of the two sets thus obtained, we see that

On any non-parabolic irreducible surface F of S_4, free from multiple curves, of order ν and rank ρ, the inflexional curve is of order $6\rho-8\nu$.

The latter result has been previously obtained by Roth† by using a limiting process, familiar in enumerative geometry, which, however, may imply some restrictions on F.

Addendum

After my Oxford lecture, Dr. E. M. Bruins, who attended it, communicated to me the following covariant expression of the primal g which cuts the surface F of § 3 in its inflexional curve.

Let us consider in S_4 two distinct points x, y, and the two primals

$$f \equiv a_x^m = 0, \qquad \phi \equiv \alpha_x^n = 0$$

intersecting in F. If x is a point of F, the line xy lies on the polar quadrics of x with respect to f and ϕ if and only if

$$a_x^{m-1}a_y \equiv v_y = 0, \qquad a_x^{m-2}a_y^2 \equiv A_y^2 = 0,$$

and

$$\alpha_x^{n-1}\alpha_y \equiv w_y = 0, \qquad \alpha_x^{n-2}\alpha_y^2 \equiv \mathfrak{A}_y^2 = 0,$$

where the symbols have obvious meanings. We have now to express (§§ 2, 3) that the line of intersection of the primes $v_y = 0$, $w_y = 0$, and a third prime, $u_y = 0$ say, cuts the quadrics $A_y^2 = 0$ and $\mathfrak{A}_y^2 = 0$ in two pairs of points having a common point. This is given by the condition

$$h \equiv PR-Q^2 = 0,$$

where

$$P \equiv (ABuvw)^2, \qquad Q \equiv (A\mathfrak{A}uvw)^2, \qquad R \equiv (\mathfrak{A}\mathfrak{B}uvw)^2.$$

* The results above were communicated in a lecture to the Oxford Mathematical Colloquium (April 12th–14th, 1950); the problem of determining Γ functionally, was put to me by Professor J. G. Semple a short time before the Colloquium. A different method for obtaining an answer would be to determine Γ as the united curve of the algebraic correspondence which associates each point P of F with its tangential points, i.e. with the residual intersections of F with the tangent plane at P.

The problem can be extended in several directions, such as that of determining the inflexional curve of a V_h of S_{3h-2} or that of determining the inflexional V_{h-1} of a V_h of S_{2h} ($h \geqslant 2$).

† L. Roth, 'Some formulae for surfaces in higher space', *Proc. Cambridge Phil. Soc.* 25 (1929), 390–406.

I now show that each of the forms P, Q, R contains, mod(f,ϕ), a factor u_x^2. We have in fact

$$P \equiv (abuc\gamma)(abud\delta)a_x^{m-2}b_x^{m-2}c_x^{m-1}d_x^{m-1}\gamma_x^{n-1}\delta_x^{n-1}.$$

Introducing the symbol c from c_x into the bracket $(abud\delta)$ we obtain, mod(f,ϕ),

$$3P \equiv (abuc\gamma)(cabd\delta)a_x^{m-2}b_x^{m-2}c_x^{m-2}d_x^{m-1}\gamma_x^{n-1}\delta_x^{n-1}u_x;$$

and now, on introducing d into the first bracket, we see that, mod(f,ϕ),

$$-12P \equiv (abcd\gamma)(abcd\delta)a_x^{m-2}b_x^{m-2}c_x^{m-2}d_x^{m-2}\gamma_x^{n-1}\delta_x^{n-1}u_x^2.$$

We obtain likewise, mod(f,ϕ),

$$-6Q \equiv (acd\alpha\gamma)(acd\alpha\delta)a_x^{m-2}b_x^{m-2}c_x^{m-2}d_x^{n-2}\gamma_x^{n-1}\delta_x^{n-1}u_x^2,$$

$$-12R \equiv (\alpha\beta\gamma\delta p)(\alpha\beta\gamma\delta q)\alpha_x^{n-2}\beta_x^{n-2}\gamma_x^{n-2}\delta_x^{n-2}p_x^{m-1}q_x^{m-1}u_x^2.$$

It follows that, mod(f,ϕ), h is divisible by u_x^4, the quotient giving the required primal

$$g \equiv (abcd\pi)(abcd\chi)a_x^{m-2}b_x^{m-2}c_x^{m-2}d_x^{m-2}\pi_x^{n-1}\chi_x^{n-1}\times$$
$$\times(\alpha\beta\gamma\delta p)(\alpha\beta\gamma\delta q)\alpha_x^{n-2}\beta_x^{n-2}\gamma_x^{n-2}\delta_x^{n-2}p_x^{m-1}q_x^{m-1}-$$
$$-4\{(abc\alpha\beta)(abc\alpha\gamma)a_x^{m-2}b_x^{m-2}c_x^{m-2}\alpha_x^{n-2}\beta_x^{n-1}\gamma_x^{n-1}\}^2 = 0,$$

which is in fact of degree $6m+6n-20$.

COLLINEARITY PROPERTIES OF SETS OF POINTS

By G. A. DIRAC (*London*)

[Received 18 April 1950]

THE following theorem was conjectured by Sylvester (**1**), and first proved by Gallai (**2**) over fifty years later:

If a finite number of distinct points in a plane are such that a line through any two of them passes through a third, then all the points lie on a line.

Gallai's theorem fails if we do not restrict ourselves to the real plane; it is not true for the points of inflexion of the cubic curve. Further, it obviously fails if the number of points may be infinite. In this work I confine myself to the extended real Euclidean plane, all points are assumed to be real, and the points mentioned in the theorems are assumed to lie in the finite part of the plane; unless otherwise stated.

The above theorem can be stated in a different form: *If a finite number of distinct points $p_1, p_2,..., p_n$ are not collinear and every pair are joined by a line, at least one of the lines so obtained contains exactly two of the points.* We may call such a line a 't-line' with respect to the system of points. Gallai's theorem then states:

A finite set of distinct non-collinear points determines at least one t-line.

This paper contains some extensions arising from this theorem and the problems which it presents, and some theorems concerning points and lines of an equally simple kind.

If the smallest number of t-lines necessarily determined by n non-collinear points, whatever their configuration, is $t(n)$, Gallai's theorem states that $t(n) \geqslant 1$ for all values of n. It seems very likely that $t(n) \geqslant [\frac{1}{2}n]$ for all n, but this has never been proved. It will be shown here that $t(n) \geqslant 3$ for all values of n.

The proof is based on the following

LEMMA. *Let a finite number of points lying in the finite part of the Euclidean plane be distributed over a set L of three or more parallel lines so that each parallel line passes through at least two of them. Then these points determine at least two non-parallel t-lines which are not in L.*

The proof of this lemma and of the corollary which follows it is an adaptation of Gallai's proof of his theorem (**2**).

Label the points $p_1, p_2, \ldots, p_n$. Choose two of the points on a line belonging to L, p_a and p_b say, and call the direction from p_a to p_b the positive direction along the line. Then any line in the plane makes a unique angle ϕ with this direction, where $0 \leqslant \phi < \pi$ and ϕ is to be measured in the anti-clockwise sense.

Now suppose each pair of the points joined by a line. Then *the set of lines so obtained contains at least two t-lines; one is that line which makes the smallest positive angle ϕ with the direction $p_a p_b$; the other is that one which makes the largest angle ϕ with this direction. These two lines are obviously not parallel.*

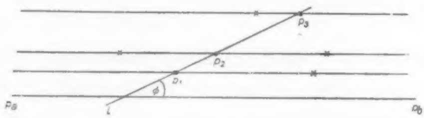

FIG. 1.

For the set of lines obtained by joining up every pair of the points $p_1, p_2, \ldots, p_n$ contains a line l making the smallest positive angle ϕ with $p_a p_b$. If it is not a t-line, suppose it passes through p_1, p_2, and p_3, p_2 lying between p_1 and p_3 on the line. The line through p_2 parallel to $p_a p_b$ passes through another point of the set, p_4 say. Then (Fig. 1) either $p_1 p_4$ or $p_3 p_4$ makes a smaller positive angle with $p_a p_b$ than l. Thus l must be a t-line.

Similarly the line making the largest angle ϕ with the direction $p_a p_b$ is a t-line. These two t-lines obviously cannot be parallel. It is implied in the lemma that the number of points is at least 6.

From this lemma we now deduce the

COROLLARY. *A finite set of distinct non-collinear points determine at least two t-lines.*

This can easily be verified for sets of less than seven points. For sets of seven or more we can use the lemma.

Join up all pairs of distinct points of the set by lines. We may assume that at least three lines pass through every point of the set; otherwise all the points would lie on two lines and, if the number of points were n, they would determine at least $n-1$ t-lines. Further, we may assume that there are points which do not lie on t-lines; otherwise the n points would determine at least $\frac{1}{2}n$ t-lines.

Project to infinity one of those points which do not lie on a t-line, keeping all the other points of the set at a finite distance. Then the

lines through this point become a set of three or more parallel lines and we obtain a system of points and lines satisfying the conditions of the lemma. Consequently in this system there must be at least two non-parallel t-lines, as in the lemma. These must be the projections of t-lines.

From the lemma and corollary we deduce the

THEOREM. *A finite set of non-collinear points determines at least three t-lines.*

In one place in the following proof we must assume that the number of points is at least eight. The least number of points necessary for the arguments to apply will always be stated, and eight is the largest number ever required. It is easy to verify the theorem for less than eight points; the following proof applies to larger sets.

Assume the theorem false, so that there are only two t-lines determined by some configuration of n points $p_1, p_2, \ldots, p_n$. The point of intersection of the two t-lines may or may not be one of the points of the set.

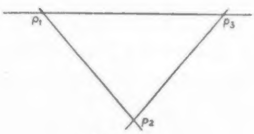

FIG. 2.

(i) Suppose the two t-lines both pass through one of the points of the set. Let this point be p_2, and let the t-lines be the lines p_1p_2 and p_2p_3. Join up all pairs of points. Every line of the resulting set of lines which passes through p_1, except p_1p_2, passes through two or more points of the set besides p_1. There must be at least three lines passing through p_1 besides p_1p_2: otherwise clearly p_1p_2 and p_2p_3 would not be the only t-lines of the system; this argument applies if we postulate two or more points besides p_1, p_2, p_3, i.e. five or more points altogether (Fig. 2).

Now project p_1p_2 to infinity. Since p_1p_2 is a t-line, all the points except p_1 and p_2 project onto the finite part of the plane. The resulting system of points and lines satisfy the conditions of the lemma, the lines through p_1 projecting into the set of parallel lines. The line p_2p_3 projects into a line passing through only one of the points concerned and can be disregarded.

It follows from the lemma that among the lines obtained after projection there must be two t-lines which are not parallel. In the original plane these correspond to lines which contain exactly two of the points $p_3, p_4, \ldots, p_n$, which do not pass through p_1, and which do not meet at a point lying on p_1p_2. These are therefore t-lines with respect to the

complete set $p_1, p_2, p_3,\ldots, p_n$ unless they pass through p_2, and so one of them must be a t-line with respect to this set.

This result contradicts the assumption that there are only two t-lines. In order that the lemma should apply, there must be at least six points in the set besides p_1 and p_2, so that the total number of points must be at least eight. But it is easy to verify that the theorem is true for all smaller sets of non-collinear points.

(ii) Suppose that the two t-lines do not intersect at one of the points of the set. Then we may assume that the two t-lines are the lines $p_1 p_2$ and $p_3 p_4$. Let them meet in the point q, which may be at infinity.

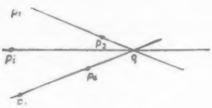

FIG. 3.

If we join all pairs of points of the set $p_1, p_2,\ldots, p_n$ by lines, at least three lines of the resulting system of lines must pass through every point p_i $(1 \leqslant i \leqslant n)$; otherwise all the points would lie on two lines and the number of t-lines would clearly be at least three. For $i \geqslant 5$ every line of the system through p_i must pass through at least two other points of the set besides p_i; otherwise it would be a t-line and $p_1 p_2$ and $p_3 p_4$ would not be the only t-lines.

Now join q to all points of the set. If q is at infinity, this means constructing lines through all the points parallel to $p_1 p_2$ and $p_3 p_4$. Either for some i, $i = n$ say, qp_i does not pass through another point of the set, or qp_i passes through another point of the set for all values of i. In the second case, qp_i passes through two other points of the set for all $i \geqslant 5$. (Fig. 3).

If qp_n does not pass through another point of the set, project it to infinity. The lines joining p_n to the other points of the set project into parallel lines, at least three in number, and each passes through two or more points of the set in the finite part of the plane. (There must be six or more points besides p_n, and accordingly the argument applies to seven or more points.) The conditions of the lemma are satisfied, so there will be two t-lines which are not parallel. The t-lines $p_1 p_2$ and $p_3 p_4$ project into parallel t-lines (they meet in q which is projected to infinity). There must therefore be a third t-line in the projection, not parallel to the other two. In the original plane this corresponds to a t-line, because it corresponds to a t-line, with respect to the points $p_1, p_2,\ldots, p_{n-1}$, which does not go through p_n. Hence in this original plane there must be three t-lines, and this contradicts the assumption that $p_1, p_2,\ldots, p_n$ determine only two t-lines.

The alternative is that qp_i passes through two other points of the set

$p_1, p_2,..., p_n$ for $i \geqslant 5$. If q is at infinity, we have a set of three or more parallel lines, each containing two or more points of the set of points in the finite part of the plane, and the conditions of the lemma therefore occur. Hence there must be two non-parallel t-lines besides $p_1 p_2$ and $p_3 p_4$. This is a contradiction. If q is not at infinity, project it to infinity, keeping the projections of $p_1, p_2,..., p_n$ at a finite distance. The preceding argument can then be repeated, leading to a contradiction. The least number of points required is seven.

This completes the proof of the theorem.

The dual of Gallai's theorem

Gallai's theorem has a dual analogue, but neither Gallai's proof (of which the proofs of the above lemma and corollary are adaptations) nor Steinberg's proof (**2**) can be dualized. It is therefore of interest to prove the dual form of Gallai's theorem directly, without appealing to the principle of duality. A proof has been given by Steenrod (**2**) using Euler's theorem on maps. I now give a more direct proof.

THEOREM. *If a finite number of distinct lines in a plane are such that through the point of intersection (possibly at infinity) of any two of them there passes a third, then all the lines are concurrent or parallel.*

Suppose the theorem false, so that a set of lines satisfy the conditions of the theorem but are not concurrent or parallel.

The points of intersection of these lines are finite in number, and so a line may be drawn which does not pass through any of them. By projecting this line to infinity, we obtain a set of lines which satisfy the conditions of the theorem and no two of which are parallel. We label these lines $l_1, l_2,..., l_n$, and their points of intersection, which are all at a finite distance, $p_1, p_2,..., p_m$.

Let (l_i, p_j), where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, denote the perpendicular distance from the point p_j to the line l_i. By hypothesis the lines are not all concurrent, and so the distances (l_i, p_j) will not all be zero. Among those which are not zero there will be at least one which does not exceed any of the others, the distance (l_1, p_1) say.

Through p_1 pass at least three lines of the set, l_2, l_3, l_4 say, none of them being parallel to l_1. Let these meet l_1 in the points p_2, p_3, p_4 respectively (Fig. 4).

Now l_2, l_3, or l_4 cannot be perpendicular to l_1. For example, if l_3 is perpendicular to l_1, then clearly, from the figure, (l_2, p_3) and (l_4, p_3) are not zero and both are smaller than (l_1, p_1), contrary to hypothesis.

If P is the foot of the perpendicular from p_1 to l_1, at least two of the points p_2, p_3, p_4 must lie on the same side of P since none can coincide with P. Suppose that p_2 and p_3 lie on the same side of P and that p_3 lies between p_2 and P on l_1. Then clearly (Fig. 4) (l_2, p_3) is positive and less than (l_1, p_1), contrary to hypothesis.

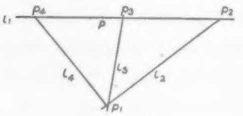

FIG. 4.

We have obtained a contradiction and the theorem is proved. This proof requires that we have four or more lines: for three the theorem is trivial.

The theorems established up to now do not permit a combinatorial formulation, they are not true in the complex projective plane (Gallai's theorem fails for the points of inflexion of the cubic curve), and so far we have confined ourselves to the extended Euclidean plane. The following result is, however, combinatorial, and applies also in the general projective plane:

Let P be a set of n non-collinear points in a plane, and let L denote the set of lines passing through at least two points of P. Then there will be a point of P such that the number of lines of L through it exceeds $\sqrt{n}$.

Suppose p_1 a point of P such that g lines of L pass through it, and suppose that the number of lines of L passing through any other point of P is not greater than g. Then the number of points of P lying on any line of L through p_1 is at most g, p_1 included; otherwise there would clearly be points of P with more than g lines of L passing through them. Thus the total number of points of P is at most $g(g-1)+1$. Hence $g(g-1)+1 \geqslant n$, i.e. $g > \sqrt{n}$. This proves the theorem.

Conclusion

If $G(n)$ is the least value of g for any configuration of n non-collinear points, we have shown that $G(n) > \sqrt{n}$.

In the first part of this essay it was shown that $t(n) \geqslant 3$. By an elaboration of the method for $t(n) \geqslant 3$ I have proved that $t(n) \geqslant 4$. The proof of this is tedious because many cases have to be considered, and it is not reproduced here.

Both these results, $t(n) \geqslant 4$ and $G(n) > \sqrt{n}$ seem to be far from best-possible in the Euclidean plane. In fact it seems likely that $t(n) \geqslant [\frac{1}{2}n]-1$ and $G(n) \geqslant [\frac{1}{2}n]$, where $[x]$ denotes the smallest integer not exceeded by x. I have checked the truth of this for $n \leqslant 14$.

In the case of $G(n)$ it is easy to see that this is best-possible, since $G(n) = [\frac{1}{2}n]$ for the following configurations: if n is even, $\frac{1}{2}n-1$ of the

points lie on one line, $\frac{1}{2}n-1$ of the remaining points lie on a second line, and these lines meet in the $(n-1)$th point; the two ranges are in perspective from the nth point. If n is odd, $\frac{1}{2}(n-1)$ of the points lie on one line, $\frac{1}{2}(n-1)$ of the others lie on a second line; these two ranges are in perspective with the nth point.

In the case of $t(n)$, $t(n) \geqslant [\frac{1}{2}n]-1$ may not be the best possible result for sufficiently large n, and the obvious case $t(n) = n-1$ might be the extreme. But it is easy to see that $t(7) = 3$ and $t(8) = 4$ and this suggests that possibly $t(n) = [\frac{1}{2}n]-1$ for infinitely many values of n.

Finally I prove the analogue of Gallai's theorem in space of three dimensions.

If a finite number of distinct points are such that no three are collinear and a plane passing through any three of them also passes through a fourth, then all the points lie in one plane.

Proof. Label the points $P_1, P_2, \ldots, P_n$. Let $\mathscr{P}$ be a plane which does not pass through any of the points, and project $P_2, P_3, \ldots, P_n$ onto $\mathscr{P}$ using P_1 as centre of perspective. Denote by p_i the intersection of the line $P_i P_1$ with $\mathscr{P}$.

Then, if $i \neq j$, $p_i \neq p_j$, and the line $p_i p_j$ passes through a third point p_k of the set of points $p_2, p_3, \ldots, p_n$. For P_1, P_i, P_j are not collinear, and the plane $P_1 P_i P_j$ must pass through a fourth point P_k. By Gallai's theorem the points $p_2, p_3, \ldots, p_n$ all lie on a line, l say, and so the points $P_1, P_2, \ldots, P_n$ all lie in the plane determined by the line l and the point P_1.

In the proof we have only assumed that among the points $P_1, P_2, \ldots, P_n$ there is one P_1 such that it is not collinear with any two of $P_2, \ldots, P_n$ and such that any plane passing through P_1 and two of $P_2, \ldots, P_n$ passes through a fourth point of the set.

The restriction that no three points should be collinear is necessary; without it the theorem fails if all the points are distributed over two skew lines.

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2. *American Mathematical Monthly*, March 1944, Problem No. 4065 by P. Erdős. [Gallai = Grünwald].

CALCULATIONS OF THE HOMOTOPY GROUPS OF A_n^2-POLYHEDRA (II)

By P. J. HILTON (*Manchester*)

[Received 2 May 1950]

1. Introduction

In the first part of this paper,† I calculated the $(n+1)$th homotopy group of an A_n^2-polyhedron in terms of its homology system $(n > 2)$. My present object is to calculate the $(n+2)$th homotopy group. This calculation depends on the nature of $\pi_{n+2}(S^n)$ $(n > 2)$. Freudenthal showed in (**7**), by means of his suspension argument,‡ that $\pi_{n+2}(S^n) \approx \pi_{n+3}(S^{n+1})$ for $n > 2$, so that it is the nature of $\pi_5(S^3)$ which is in question. It is assumed throughout this paper that $\pi_5(S^3)$ is cyclic of order 2.§ The calculation of $\pi_{n+2}(A_n^2)$ also depends on the nature of $\pi_{n+2}(e^{n+1} \cup S^n)$, where e^{n+1} is attached to S^n by a map of even degree σ. It is proved that this group contains $\pi_{n+2}(S^n)$ as a subgroup of index 2, and the group is calculated if σ is divisible by 4. In the main body of the paper, I shall assume $n > 3$, and the special case $n = 3$ will be reserved for a separate section at the end of the paper.

For convenience I recall here the specifications for a typical reduced A_n^2-polyhedron. It is a cell-complex K, of at most $n+2$ dimensions, such that

$K^0 = K^1 = ... = K^{n-1} = e^0$, a single point;||

$K^n = S_1^n \cup ... \cup S_m^n$, a bundle of n-spheres with the single common point e^0;

$K^{n+1} = K^n \cup e_1^{n+1} \cup ... \cup e_{t+l}^{n+1}$, where e_i^{n+1} $(1 \leqslant i \leqslant t \leqslant m)$ is attached to K^n by a map f_i: $\dot{E}_i^{n+1} \to S_i^n$ of degree σ_i, σ_i being odd if and only if $1 \leqslant i \leqslant h \leqslant t$, and e^0 closes e_{t+j}^{n+1} to a sphere S_j^{n+1} $(1 \leqslant j \leqslant l)$;

$K^{n+2} = K^{n+1} \cup e_1^{n+2} \cup ... \cup e_{p+u}^{n+2}$, where e_i^{n+2} $(1 \leqslant i \leqslant p)$ is attached to K^{n+1} by a map g_i: $\dot{E}_i^{n+2} \to \sum \gamma_{ij} S_j^n$; e_{p+i}^{n+2} $(1 \leqslant i \leqslant u-k)$ is attached to K^{n+1} by a map g_{p+i}: $\dot{E}_{p+i}^{n+2} \to S_{k+i}^{n+1} \cup \sum \gamma_{p+i,j} S_j^n$, which is of even

† See (**1**). In (**2**) J. H. C. Whitehead described as an A_n^2-polyhedron a finite connected cell-complex of at most $(n+2)$ dimensions whose first $n-1$ homotopy groups vanished.

‡ 'Suspension' is the English translation of Freudenthal's 'Einhängung'.

§ J. H. C. Whitehead has informed me that this has been proved independently by G. W. Whitehead and L. Pontrjagin, though the contrary result was announced in (**8**). [See G. W. Whitehead, *Annals of Math.* 52 (1950), 245–7.]

|| K^r stands for the r-dimensional skeleton of K, i.e. the collection of those cells of K whose dimension does not exceed r.

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degree τ_{k+i} over S^{n+1}_{k+i}; and $e^{n+2}_{p+u-k+i}$ $(1 \leqslant i \leqslant k)$ is attached to K^{n+1} by a map $g_{p+u-k+i}\colon \dot{E}^{n+2}_{p+u-k+i} \to S^{n+1}_i$ of odd degree τ_i.

In these specifications $\gamma_{ij} = 0$ or 1, $\gamma_{ij} S^n_j$ is to be understood as S^n_j if $\gamma_{ij} = 1$ and as e^0 if $\gamma_{ij} = 0$, so that $\sum \gamma_{ij} S^n_j$ stands for the union of those n-spheres S^n_j for which $\gamma_{ij} = 1$, unless $\gamma_{ij} = 0$ for all i, when it stands for e^0. The maps g_i $(1 \leqslant i \leqslant p+u-k)$ are to be understood as essential over those n-spheres S^n_j for which $\gamma_{ij} = 1$. The index j ranges over the values $h+1,\ldots, m$, and the γ_{ij} are, in fact, the coefficients in the postulated homomorphism† $\gamma\colon H_{n+2}(K, 2) \to H_n(K, 2)$.

2. Two preliminary lemmas

The following two lemmas, together with the theorems in (**5**), effectively complete the machinery for calculating $\pi_{n+2}(K)$ $(n > 3)$.

LEMMA 2.1. *Let X, Y be two (arcwise-connected) topological spaces with a single point x_0 in common, and let $\pi_r(X \cup Y) = i_r \pi_r(X) + j_r \pi_r(Y)$, where i_r, j_r are the injection isomorphisms induced by the identity maps $X \to X \cup Y$, $Y \to X \cup Y$. Further let a t-cell $(r < 2t-1)$ be attached to X (and disjoint from Y) to form X^*. Finally let $X \cup Y$ be aspherical in the first $(r-t+1)$ dimensions. Then*

$$\pi_r(X^* \cup Y) = i^*_r \pi_r(X^*) + j^*_r \pi_r(Y),$$

where i^*_r, j^*_r are the injection isomorphisms induced by the identity maps $X^* \to X^* \cup Y$, $Y \to X^* \cup Y$.

This lemma is a special case of Theorem 3 of (**10**). It also follows as an easy consequence of Theorem 8 of (**3**).

LEMMA 2.2. *If G is an abelian group, and H is a subgroup of G such that the difference group $G-H$ is a free cyclic group, then G is the direct sum of H and a free cyclic group.*

This lemma is a special case of a standard algebraical theorem on central extensions of free abelian groups.

3. $\pi_{n+2}(K)$ $(n > 3)$

I now assume $n > 3$. I recall that we are taking $\pi_{n+2}(S^n)$ to be cyclic of order 2. Its essential class is the class containing the map

$$S^{n+2} \to S^{n+1} \to S^n,$$

where $S^{n+2} \to S^{n+1}$ and $S^{n+1} \to S^n$ are essential maps. For the suspension homomorphism $E\colon \pi_4(S^2) \to \pi_5(S^3)$ is onto, and the essential class in $\pi_4(S^2)$ is that containing the map $S^4 \to S^3 \to S^2$, where $S^4 \to S^3$ is essential and $S^3 \to S^2$ is the Hopf map with Hopf invariant 1.

† See (**1**), where the homology system of K is defined to include γ.

Now $\pi_{n+2}(K^n) \approx \pi_{n+2}(S_1^n)+...+\pi_{n+2}(S_m^n)$. For Theorem 3*b* of (**5**) asserts that $\pi_{n+2}(X \cup S^n) \approx \pi_{n+2}(X)+\pi_{n+2}(S^n)$, if X is attached to S^n at a single point and $\pi_s(X) = 0$ $(1 \leqslant s \leqslant 3)$. Thus, by an obvious induction on m, we get

$$\pi_{n+2}(K^n) = \pi_{n+2}(S_1^n \cup ... \cup S_m^n) \approx \pi_{n+2}(S_1^n)+...+\pi_{n+2}(S_m^n). \tag{3.1}$$

Let us write $\pi_{n+2}(K^n) \approx (\bar{a}_1,..., \bar{a}_m)$, where $\bar{a}_i$ is the class of the map of S^{n+2} onto S_i^n which is compounded of essential maps $S^{n+2} \to S^{n+1}$ and $S^{n+1} \to S_i^n$.

By a further application of Theorem 3*b* of (**5**), it follows that

$$\pi_{n+2}(K^n \cup S_1^{n+1} \cup ... \cup S_l^{n+1}) \approx (\bar{a}_1,..., \bar{a}_m)+\pi_{n+2}(S_1^{n+1})+...+\pi_{n+2}(S_l^{n+1}).$$

Each of the groups $\pi_{n+2}(S_i^{n+1})$ is cyclic of order 2. Let the (free abelian) group $H_{n+1}(K^{n+1})$ be written $(b_1^*,..., b_l^*)$, where b_i^* corresponds to the cycle e_{t+i}^{n+1} $(i = 1,..., l)$. Writing K_1^{n+1} for $K^n \cup S_1^{n+1} \cup ... \cup S_l^{n+1}$, we have

$$\pi_{n+2}(K_1^{n+1}) \approx (\bar{a}_1,..., \bar{a}_m)+(\bar{b}_1^*,..., \bar{b}_l^*), \tag{3.2}$$

where $\bar{b}_i^*$ is the residue class (mod 2) of b_i^*.

Repeated application of Lemma 2.1 entitles us to complete the calculation of $\pi_{n+2}(K^{n+1})$ by the replacement of the direct summand $(\bar{a}_1,..., \bar{a}_t)$ by t direct summands, respectively isomorphic to $\pi_{n+2}(S_i^n \cup e_i^{n+1})$, where e_i^{n+1} is attached to S_i^n by a map f_i: $\dot{E}_i^{n+1} \to S_i^n$ of degree σ_i $(i = 1,..., t)$. We use the notation of (**1**), so that†

$$\pi_{n+2}(S_i^n \cup e_i^{n+1})-i_{n+2}\,\pi_{n+2}(S_i^n) \approx h_{n+1}^{-1}(0), \tag{3.3}$$

where i_{n+2}: $\pi_{n+2}(S_i^n) \to \pi_{n+2}(S_i^n \cup e_i^{n+1})$ is induced by the identity map $S_i^n \to S_i^n \cup e_i^{n+1}$, h_r: $\pi_r(\dot{E}_i^{n+1}) \to \pi_r(S_i^n)$ is induced by f_i, and

$$i_{n+2}^{-1}(0) = h_{n+2}\,\pi_{n+2}(\dot{E}_i^{n+1}). \tag{3.4}$$

We distinguish two cases, (i) σ_i odd, (ii) σ_i even, and, for convenience, we drop the suffix i.

LEMMA 3.5. *If σ is odd, $\pi_{n+2}(S^n \cup e^{n+1}) = 0$.*

For, since f is of odd degree, $h_s\,\pi_s(\dot{E}^{n+1}) = \pi_s(S^n)$ $(s = n+1,\ n+2)$. Thus $h_{n+1}^{-1}(0) = 0$ and $i_{n+2}^{-1}(0) = \pi_{n+2}(S^n)$. The lemma now follows from (3.3).

LEMMA 3.6. *If σ is even, $\pi_{n+2}(S^n \cup e^{n+1})$ is an extension of a cyclic group of order 2 by a cyclic group of order 2. It is the direct sum of two cyclic groups of order 2 if σ is divisible by 4. If σ is of the form $2r$, where r is odd, then $\pi_{n+2}(S^n \cup e^{n+1})$ is independent of r.*

† See (**3**), Theorem 8. Theorem 1 of (**5**) is a reformulation (and generalization) of Theorem 8 of (**3**), using the apparatus of relative homotopy groups.

Since f is of even degree, $h_s \pi_s(\dot{E}^{n+1}) = 0$ $(s = n+1,\ n+2)$. Thus $h_{n+1}^{-1}(0) = \pi_{n+1}(\dot{E}^{n+1})$ and $i_{n+2}^{-1}(0) = 0$. Since $\pi_{n+1}(\dot{E}^{n+1})$ is cyclic of order 2, $\pi_{n+2}(S^n \cup e^{n+1})$, being an extension† of $h_{n+1}^{-1}(0)$ by $i_{n+2}\pi_{n+2}(S^n)$, is an extension of a cyclic group of order 2 by a cyclic group of order 2.

Now let $\sigma = 2r$, where r is even. Suppose e'^{n+1} attached to S'^n by a map $f'\colon \dot{E}^{n+1} \to S'^n$ of degree r. Let $p_0\colon S'^n \to S^n$ be a map of degree 2. Then we may assume that $e^{n+1} \cup S^n$ is defined by identifying, in $e'^{n+1} \cup S'^n$, each $x \in S'^n$ with $p_0 x \in S^n$.

Let $\phi'\colon (E^{n+1}, \dot{E}^{n+1}) \to (e'^{n+1} \cup S'^n, S'^n)$ be the characteristic map for e'^{n+1}, so that $\phi' \mid \dot{E}^{n+1} = f'$, and let $p\colon (e'^{n+1} \cup S'^n, S'^n) \to (e^{n+1} \cup S^n, S^n)$ be the transformation given by

$$px = x, \quad x \in e'^{n+1}, \qquad p \mid S'^n = p_0.$$

Then $p\phi'$ is the characteristic map for e^{n+1}. Let F be a closed set in $e^{n+1} \cup S^n$. Then, since ϕ' maps E^{n+1} onto $e'^{n+1} \cup S'^n$,

$$p^{-1}(F) = \phi'(p\phi')^{-1}(F).$$

Since E^n is compact, this shows that p is continuous.

The map p induces homomorphisms

$$\mu\colon \pi_{n+2}(e'^{n+1} \cup S'^n) \to \pi_{n+2}(e^{n+1} \cup S^n),$$

$$\nu\colon \pi_{n+2}(e'^{n+1} \cup S'^n, S'^n) \to \pi_{n+2}(e^{n+1} \cup S^n, S^n),$$

and the map ϕ' induces the homomorphism

$$g\colon \pi_{n+2}(E^{n+1}, \dot{E}^{n+1}) \to \pi_{n+2}(e'^{n+1} \cup S'^n, S'^n).$$

Consider the diagram

$$\begin{array}{ccccc} & & & & \pi_{n+2}(E^{n+1}, \dot{E}^{n+1}) \\ & & & & \downarrow g \\ \pi_{n+2}(S'^n) & \xrightarrow{i'} & \pi_{n+2}(e'^{n+1} \cup S'^n) & \xrightarrow{j'} & \pi_{n+2}(e'^{n+1} \cup S'^n, S'^n) \\ & & \downarrow \mu & & \downarrow \nu \\ \pi_{n+2}(S^n) & \xrightarrow{i} & \pi_{n+2}(e^{n+1} \cup S^n) & \xrightarrow{j} & \pi_{n+2}(e^{n+1} \cup S^n, S^n). \end{array}$$

The homomorphisms i, j, i', j' are the homomorphisms of the homotopy sequence, and g, νg are induced by the characteristic maps for e'^{n+1}, e^{n+1}. Thus, by Theorem 1 of (**5**), g and νg are isomorphisms onto, so that ν is an isomorphism onto. Let $b' \in \pi_{n+2}(e'^{n+1} \cup S'^n)$ be such that $j'b' \neq 0$. Since $\pi_{n+2}(e'^{n+1} \cup S'^n)$ is an extension of a cyclic group of order 2 by $i'\pi_{n+2}(S'^n)$, this means that $2b' = 0$ or $i'a'$, where a' generates $\pi_{n+2}(S'^n)$. Since ν is an isomorphism, $\nu j'b' \neq 0$, and, since j, j', μ, ν are connected by the equation $j\mu = \nu j'$, it follows that $j\mu b' \neq 0$. Thus $\mu b'$ is an element of $\pi_{n+2}(e^{n+1} \cup S^n)$ which is not in $i\pi_{n+2}(S^n)$. Now $\mu i'a' = 0$,

† I follow Eilenberg–Maclane in saying that G is an extension of H by Z if there exists a homomorphism of G onto H with kernel Z.

since p_0 is a map of even degree, so that $2\mu b' = \mu(2b') = 0$. Thus $\pi_{n+2}(e^{n+1} \cup S^n)$ contains two distinct elements of order 2, which implies that it is the direct sum of two cyclic groups of order 2.

Now let $\sigma = 2r$, where r is odd. Suppose now that e'^{n+1} is attached to S'^n by a map of $f'\colon \dot{E}^{n+1} \to S'^n$ of degree 2, and let

$$p\colon e'^{n+1} \cup S'^n \to e^{n+1} \cup S^n$$

be defined as above, except that $p_0\colon S'^n \to S^n$ is now a map of degree r. The arguments used above are again valid as far as the assertion that $\mu b'$ is an element of $\pi_{n+2}(e^{n+1} \cup S^n)$ which is not in $i\pi_{n+2}(S^n)$. Since p_0 is now of odd degree, $\mu i'a' = ia$, where a generates $\pi_{n+2}(S^n)$. Thus $2\mu b' = 0$ if $2b' = 0$, and $2\mu b' = ia$ if $2b' = i'a'$. This completes the proof of the lemma.

From these two lemmas we are able to conclude that

$$\pi_{n+2}(K^{n+1}) \approx P_n(\sigma_{h+1})+\ldots+P_n(\sigma_t)+(\bar{a}_{t+1},\ldots,\bar{a}_m)+(\bar{b}_1^*,\ldots,\bar{b}_l^*), \quad (3.7)$$

where $P_n(\sigma) = \pi_{n+2}(e^{n+1} \cup S^n)$, e^{n+1} being attached to S^n by a map of even degree σ. It should be recalled that $P_n(\sigma)$ contains $\pi_{n+2}(S^n)$ as a subgroup of index 2, so that $P_n(\sigma_{h+1})+\ldots+P_n(\sigma_t)$ contains $(\bar{a}_{h+1},\ldots, \bar{a}_t)$ as a subgroup of index 2^{t-h}.

Now attach e_1^{n+2} to K^{n+1}. Then

$$\pi_{n+2}(K^{n+1} \cup e_1^{n+2}) - i_{n+2}\,\pi_{n+2}(K^{n+1}) \approx h_{n+1}^{-1}(0),$$

where $h_r\colon \pi_r(\dot{E}_1^{n+2}) \to \pi_r(K^{n+1})$ is induced by $g_1\colon \dot{E}_1^{n+2} \to K^{n+1}$ and i_{n+2} is, as usual, the injection homomorphism. We recall from (**1**) that

$$\pi_{n+1}(K^{n+1}) \approx \pi_{n+1}(S_{h+1}^n)+\ldots+\pi_{n+1}(S_m^n)+\pi_{n+1}(S_1^{n+1})+\ldots+\pi_{n+1}(S_l^{n+1}).$$

It follows that h_{n+1} maps $2\pi_{n+1}(\dot{E}_1^{n+2})$ into zero. Since $2\pi_{n+1}(\dot{E}_1^{n+2})$ is maximal in $\pi_{n+1}(\dot{E}_1^{n+2})$ and $2\pi_{n+1}(\dot{E}_1^{n+2}) \approx \pi_{n+1}(\dot{E}_1^{n+2})$, it follows that $h_{n+1}^{-1}(0) \approx \pi_{n+1}(\dot{E}_1^{n+2})$ and is cyclic infinite. Thus, by Lemma 2.2,

$$\pi_{n+2}(K^{n+1} \cup e_1^{n+2}) \approx h_{n+2}\,\pi_{n+2}(K^{n+1})+\pi_{n+1}(\dot{E}_1^{n+2}).$$

Now $i_{n+2}^{-1}(0) = h_{n+2}\,\pi_{n+2}(\dot{E}_1^{n+2})$, which is the cyclic subgroup of $\pi_{n+2}(K^{n+1})$ generated by $\sum \gamma_{1j}\bar{a}_j$. Thus $\pi_{n+2}(K^{n+1} \cup e_1^{n+2})$ is obtained from

$$\pi_{n+2}(K^{n+1})+\pi_{n+1}(\dot{E}_1^{n+2})$$

by adding the relations $\sum \gamma_{1j}\bar{a}_j = 0$.

Now attach e_2^{n+2}. Since $\pi_{n+1}(K^{n+1} \cup e_1^{n+2})$ is obtained from $\pi_{n+1}(K^{n+1})$ by adding a relation, it follows as above that

$$\pi_{n+2}(K^{n+1} \cup e_1^{n+2} \cup e_2^{n+2}) \approx i_{n+2}\,\pi_{n+2}(K^{n+1} \cup e_1^{n+2})+\pi_{n+1}(\dot{E}_2^{n+2}),$$

and $i_{n+2}^{-1}(0)$ is generated by $\sum \gamma_{2j}\bar{a}_j$, where $\bar{a}_j$ $(j = h+1,\ldots, m)$ are now generators of $\pi_{n+2}(K^{n+1} \cup e_1^{n+2})$. Thus $\pi_{n+2}(K^{n+1} \cup e_1^{n+2} \cup e_2^{n+2})$ is

obtained from $\pi_{n+2}(K^{n+1})+\pi_{n+1}(\dot{E}_1^{n+2})+\pi_{n+1}(\dot{E}_2^{n+2})$ by adding the relations $\sum \gamma_{1j}\bar{a}_j = 0$, $\sum \gamma_{2j}\bar{a}_j = 0$.

Proceeding in this way for each of the $(n+2)$-cells $e_1^{n+2},\ldots, e_p^{n+2}$, and writing $K_1^{n+2} = K^{n+1} \cup e_1^{n+2} \cup \ldots \cup e_p^{n+2}$, we see that $\pi_{n+2}(K_1^{n+2})$ is obtained from $\pi_{n+2}(K^{n+1})+\pi_{n+1}(\dot{E}_1^{n+2})+\ldots+\pi_{n+2}(\dot{E}_p^{n+2})$ by adding the relations $\sum \gamma_{ij}\bar{a}_j = 0$ $(i = 1,\ldots, p)$. Now $H_{n+2}(K)$ is a free abelian group on p generators, corresponding to the cells $e_1^{n+2},\ldots, e_p^{n+2}$. Thus we may identify $\pi_{n+1}(\dot{E}_1^{n+2})+\ldots+\pi_{n+1}(\dot{E}_p^{n+2})$ with $H_{n+2}(K)$, so that $\pi_{n+2}(K_1^{n+2})$ is obtained from $\pi_{n+2}(K^{n+1})+H_{n+2}(K)$ by adding the relations $\sum \gamma_{ij}\bar{a}_j = 0$ $(i = 1,\ldots, p)$.

Now attach e_{p+1}^{n+2}. Then $\pi_{n+2}(K_1^{n+2} \cup e_{p+1}^{n+2}) - i_{n+2}\pi_{n+2}(K_1^{n+2}) \approx h_{n+1}^{-1}(0)$, where $h_r\colon \pi_r(\dot{E}_{p+1}^{n+2}) \to \pi_r(K_1^{n+2})$ is induced by the attaching map $g_{p+1}\colon \dot{E}_{p+1}^{n+2} \to K^{n+1}$, followed by the identity map $K^{n+1} \to K_1^{n+2}$. Now $\pi_{n+1}(S_{k+1}^{n+1})$ is embedded isomorphically in $\pi_{n+1}(K_1^{n+2})$ as a direct summand, and g_{p+1} is of positive degree τ_{k+1} over S_{k+1}^{n+1}. This shows that any non-zero element of $\pi_{n+1}(\dot{E}_{p+1}^{n+2})$ is mapped by h_{n+1} into a non-zero element of $\pi_{n+1}(K_1^{n+2})$, or that $h_{n+1}^{-1}(0) = 0$. On the other hand, since τ_{k+1} is even, $h_{n+2}\pi_{n+2}(\dot{E}_{p+1}^{n+2})$ is the cyclic subgroup of $\pi_{n+2}(K_1^{n+2})$ generated by $\sum \gamma_{p+1,j}\bar{a}_j$. Arguing similarly for each of the $(n+2)$-cells $e_{p+1}^{n+2},\ldots, e_{p+u-k}^{n+2}$, and writing K_2^{n+2} for $K_1^{n+2} \cup e_{p+1}^{n+2} \cup \ldots \cup e_{p+u-k}^{n+2}$, we have the result that $\pi_{n+2}(K_2^{n+2})$ is obtained from $\pi_{n+2}(K^{n+1})+H_{n+2}(K)$ by adding the relations $\sum \gamma_{ij}\bar{a}_j = 0$ $(i = 1,\ldots, p+u-k)$. If we identify $\bar{a}_j$ with the corresponding generator of $H_n(K,2)$, and write†

$$H_{n+2}(K,2) = \mu H_{n+2}(K)+\Delta_2^* H_{n+1}(K) = (\bar{c}_1,\ldots,\bar{c}_p,\bar{c}_{p+1},\ldots,\bar{c}_{p+u-k}),$$

then $\gamma\colon H_{n+2}(K,2) \to H_n(K,2)$ is given by

$$\gamma\bar{c}_i = \sum \gamma_{ij}\bar{a}_j \quad (i = 1,\ldots, p+u-k).$$

We then have

$$\pi_{n+2}(K_2^{n+2}) \approx (\pi_{n+2}(K^{n+1})-\gamma H_{n+2}(K,2))+H_{n+2}(K). \tag{3.8}$$

Now attach $e_{p+u-k+1}^{n+2}$. Then

$$\pi_{n+2}(K_2^{n+2} \cup e_{p+u-k+1}^{n+2}) - i_{n+2}\pi_{n+2}(K_2^{n+2}) \approx h_{n+1}^{-1}(0),$$

where $h_r\colon \pi_r(\dot{E}_{p+u-k+1}^{n+2}) \to \pi_r(K_2^{n+2})$ is induced by

$$g_{p+u-k+1}\colon \dot{E}_{p+u-k+1}^{n+2} \to K^{n+1},$$

followed by the identity map $K^{n+1} \to K_2^{n+2}$. Since $\pi_{n+1}(S_1^{n+1})$ is embedded isomorphically in $\pi_{n+1}(K_2^{n+2})$ as a direct summand, it follows as

† See (1), (2). The homomorphism μ is the natural homomorphism

$$H_r(K) \to H_r(K)-2H_r(K),$$

and Δ^* is a certain isomorphism of ${}_2H_{n+1}(K)$ into $H_{n+2}(K,2)$.

above that $h_{n+1}^{-1}(0) = 0$. On the other hand, since $g_{p+u-k+1}$ is a map of odd degree over S_1^{n+1}, $h_{n+2}\,\pi_{n+2}(\dot{E}_{p+u-k+1}^{n+2}) = \pi_{n+2}(S_1^{n+1})$. Thus the effect of attaching $e_{p+u-k+1}^{n+2}$ is to annihilate the generator $\bar{b}_1^*$. Similarly, the effect of attaching the remaining $(n+2)$-cells is to annihilate the generators $(\bar{b}_1^*,..., \bar{b}_k^*)$. Now each generator $\bar{b}_i^*$ $(i = k+1,..., l)$ may be identified with the corresponding generator in $\mu H_{n+1}(K)$. With this identification, and recalling (3.7), we see that

$$\pi_{n+2}(K) \approx \mu H_{n+1}(K) + H_{n+2}(K) + \\ + \Big\{\Big(\sum_{i=h+1}^{t} P_n(\sigma_i) + (\bar{a}_{t+1},..., \bar{a}_m)\Big) - \gamma H_{n+2}(K,2)\Big\}. \quad (3.9)$$

Now $\sum_{i=h+1}^{t} P_n(\sigma_i) + (\bar{a}_{t+1},..., \bar{a}_m)$ is an extension of a free mod 2 module of rank $(t-h)$ by $H_n(K,2)$. Moreover the free mod 2 module arises from those n-cells $e_{h+1}^n,..., e_t^n$ which bound when taken an even number of times. It may thus be identified with ${}_2H_n(K)$. This shows that the group $\Big(\sum_{i=h+1}^{t} P_n(\sigma_i) + (\bar{a}_{t+1},..., \bar{a}_m)\Big) - \gamma H_{n+2}(K,2)$ is an extension of ${}_2H_n(K)$ by $H_n(K,2) - \gamma H_{n+2}(K,2)$. We have proved

THEOREM 3.10. *Let $n > 3$, and let K be an A_n^2-polyhedron. Then the $(n+2)$-th homotopy group of K is given in terms of its homology system by the isomorphism*

$$\pi_{n+2}(K) \approx \mu H_{n+1} + H_{n+2} + E,$$

where E is an extension of ${}_2H_n$ by $H_n(2) - \gamma H_{n+2}(2)$.

We know from Lemma 3.6 that $P_n(\sigma_i)$ is the direct sum of two cyclic groups of order 2 if σ_i is divisible by 4. We therefore have

COROLLARY 3.11. *If, in addition, K has no n-dimensional torsion coefficients of the form $4r+2$, then*†

$$\pi_{n+2}(K) \approx (H_n(2) - \gamma H_{n+2}(2)) + {}_2H_n + \mu H_{n+1} + H_{n+2}.$$

Finally, we investigate more closely the nature of the extension E. There are two possibilities, depending on the value of $P_n(2)$. If $P_n(2)$ is the direct sum of two cyclic groups of order 2, then the conclusion of Corollary 3.11 holds without the restriction on K. If $P_n(2)$ is cyclic of order 4, the situation is more complicated.

Let H_n be given as $(a_1,..., a_m)$, where a_i is of order σ_i if $i \leqslant t$, and a_j is of infinite order when $t+1 \leqslant j \leqslant m$. Further let σ_i be odd if and only if $i \leqslant h \leqslant t$. Then ${}_2H_n = (\frac{1}{2}\sigma_{h+1}a_{h+1},..., \frac{1}{2}\sigma_t a_t)$, and $H_n(2) = (\bar{a}_{h+1},..., \bar{a}_m)$.

† The group $H_n(2) - \gamma H_{n+2}(2)$ is isomorphic to its dual group $\gamma^{*-1}(0)$, where $\gamma^*: H^n(2) \to H^{n+2}(2)$ is the Steenrod square. See (**2**), (**6**).

To determine the nature of the extension E, it is only necessary to choose representatives in E for $(\frac{1}{2}\sigma_{h+1}a_{h+1},..., \frac{1}{2}\sigma_t a_t)$, say $(\alpha_{h+1},..., \alpha_t)$, and to determine the nature of the elements $2\alpha_{h+1},..., 2\alpha_t$.

If $P_n(2)$ is cyclic of order 4, then the extension E is determined by the relations

$$\left.\begin{array}{ll} 2\alpha_i = 0, & \text{if } \sigma_i \text{ is divisible by } 4 \\ 2\alpha_i = \theta\bar{a}_i, & \text{otherwise} \end{array}\right\} \quad (i = h+1,..., t), \tag{3.12}$$

where θ is the natural homomorphism $\theta: H_n(2) \to H_n(2)-\gamma H_{n+2}(2)$.

4. The case $\pi_{n+1}(K) = 0$

Let us assume that $\gamma: H_{n+2}(2) \to H_n(2)$ is a homomorphism onto. Then $H_n(2)-\gamma H_{n+2}(2) = 0$, and it may be seen that

$$\pi_{n+2}(K) \approx {}_2H_n+\mu H_{n+1}+H_{n+2}.$$

The interest of this case arises from the fact that, if $\pi_{n+1}(K) = 0$, then γ is onto. To see this, recall that $\pi_{n+1}(K)$ is an extension of $H_{n+1}(K)$, so that, if $\pi_{n+1}(K) = 0$, $H_{n+1}(K) = 0$. Then it follows from (**1**) that $\pi_{n+1}(K)$ is got from $H_n(2)$ by adding the relations $\gamma\bar{c}_i = 0$ $(i = 1,..., p)$, where $H_{n+2}(2) = (\bar{c}_1,..., \bar{c}_p)$. Since $\pi_{n+1}(K) = 0$, this means that† $\gamma H_{n+2}(2) = H_n(2)$. We therefore have the theorem:

THEOREM 4.1. *Let $n > 3$, and let K be an A_n^2-polyhedron such that $\pi_{n+1}(K) = 0$. Then*

$$\pi_{n+2}(K) \approx {}_2H_n+H_{n+2}.$$

A model for an A_n^2-cohomology system yielding $\pi_{n+1} = 0$ was constructed in (**2**).

5. The case $n = 3$

The methods used in this paper are based essentially on Theorem 8 of (**3**) and Theorems 1 and 3*b* of (**5**). These theorems do not yield $\pi_s(K^3)$, where K is a reduced A_3^2-polyhedron. However, S. C. Chang has shown in (**10**) that, if $\pi_r(X)$ vanishes for $r = 1,..., m-n$, where $m < 2n$, and if an n-sphere S^n is attached to a single point of X, then

$$\pi_m(X \cup S^n) \approx \pi_m(X)+\pi_m(S^n)+[\pi_{m-n+1}(X), \pi_n(S^n)], \tag{5.1}$$

where the last term is the Whitehead product as defined in (**4**). This result, which extends Theorem 3*b* of (**5**), may also be regarded as a generalization of Theorem 2 of (**4**).

† This was also shown in (**2**) without using the explicit nature of $\pi_{n+1}(K)$. It is well known that, if $\pi_s(X) = 0$ $(s = 1,..., n-1)$, then the natural homomorphism of $\pi_{n+1}(X)$ into $H_{n+1}(X)$ is onto. See (**9**).

Further, Theorems $8b$ and $8c$ of (**3**), which have been used very frequently in the rest of this paper, are inapplicable to the calculation of $\pi_5(K^4)$. Again, S. C. Chang has shown in (**10**) that, if $m < 2n-2$; $\pi_s(X)$ vanishes for $s = 1,\dots, m-n+1$; if e^n is attached to X by a map $f: \dot{E}^n \to X$ which induces $h_r: \pi_r(\dot{E}^n) \to \pi_r(X)$; and if $i_m: \pi_m(X) \to \pi_m(X \cup e^n)$ is induced by the identity map $X \to X \cup e^n$, then†

$$i_m^{-1}(0) = \{h_m \pi_m(\dot{E}^n), [h_{n-1}\pi_{n-1}(\dot{E}^n), \pi_{m-n+2}(X)]\}. \tag{5.2}$$

This result extends Theorem $8c$ of (**3**).

Theorem $8a$ of (**3**) is still available, asserting that the homomorphism‡ of $h_{m-1}^{-1}(0)$ into $\pi_m(X \cup e^n) - i_m \pi_m(X)$ is onto. However, the kernel of the homomorphism remains to be calculated.

We now consider the application of these results to the calculation of $\pi_5(K)$, where K is an A_3^2-polyhedron. I first prove two lemmas, similar in nature to Lemmas 3.5 and 3.6.

LEMMA 5.3. *Let e^4 be attached to S^3 by a map of odd degree. Then $\pi_5(e^4 \cup S^3) = 0$.*

Now h_4 maps $\pi_4(\dot{E}^4)$ onto $\pi_4(S^3)$, so that $h_4^{-1}(0) = 0$; but there is a homomorphism of $h_4^{-1}(0)$ onto $\pi_5(e^4 \cup S^3) - i_5 \pi_5(S^3)$, so that

$$\pi_5(e^4 \cup S^3) = i_5 \pi_5(S^3).$$

By (5.2), $i_5^{-1}(0) \supset h_5 \pi_5(\dot{E}^4) = \pi_5(S^3)$, so that $i_5 \pi_5(S^3) = 0$ and the lemma is proved.

LEMMA 5.4. *Let e^4 be attached to S^3 by a map of even degree, σ. Then $\pi_5(e^4 \cup S^3)$ is an extension of a cyclic group of order 2 by a cyclic group of order 2. It is the direct sum of two cyclic groups of order 2 if σ is divisible by 4. If σ is of the form $2r$, where r is odd, then $\pi_5(e^4 \cup S^3)$ is independent of r.*

Now h_4 maps $\pi_4(\dot{E}^4)$ onto 0, so that $h_4^{-1}(0) = \pi_4(\dot{E}^4)$, and there is a homomorphism of $\pi_4(\dot{E}^4)$ onto $\pi_5(e^4 \cup S^3) - i_5 \pi_5(S^3)$. Moreover,

$$i_5^{-1}(0) = \{h_5 \pi_5(\dot{E}^4), [h_3 \pi_3(\dot{E}^4), \pi_3(S^3)]\},$$

by (5.2). However, since twice any element in $\pi_5(S^3)$ is 0, $i_5^{-1}(0) = 0$, and i_5 is an isomorphism. Thus $\pi_5(e^4 \cup S^3) - i_5 \pi_5(S^3)$ is 0 or cyclic of order 2, and we dispose of the first possibility.

† If A, B are subgroups of the abelian group G, then $\{A, B\}$ stands for the subgroup of G generated by them.

‡ In considering the attachment of a 4-cell to S^3, we have $m = 5$, $n = 4$, so that $m < 2n-1$ and $\pi_s(S^3) = 0$, $s = 1,\dots, m-n+1$ $(= 2)$.

Assume then that $\pi_5(e^4 \cup S^3) = i_5 \pi_5(S^3)$. This implies that $\pi_5(e^4 \cup S^3)$ is cyclic of order 2 and that any map of S^5 into $e^4 \cup S^3$ is homotopic to a map of S^5 into S^3. Attach another cell e^{*4} to S^3, also by a map of degree σ. The resulting complex is clearly of the same homotopy type as $e^4 \cup S^3 \cup S^4$, where S^4 is attached to $e^4 \cup S^3$ at a single point of S^3. Thus $\pi_5(e^{*4} \cup e^4 \cup S^3) \approx \pi_5(e^4 \cup S^3 \cup S^4) \approx \pi_5(e^4 \cup S^3) + \pi_5(S^4)$, and is therefore, by hypothesis, the direct sum of two cyclic groups of order 2. Let $h_r^*\colon \pi_r(\dot{E}^{*4}) \to \pi_r(e^4 \cup S^3)$ be induced by the attaching map for e^{*4}. Then $h_4^{*-1}(0) = \pi_4(\dot{E}^{*4})$, and there is a homomorphism of $\pi_4(\dot{E}^{*4})$ onto $\pi_5(e^{*4} \cup e^4 \cup S^3) - i_5 \pi_5(e^4 \cup S^3)$. This latter difference group is non-zero. Moreover, the homomorphism is achieved by constructing, from a map of S^4 into $\dot{E}^{*4}$, a map of S^5 into $e^{*4} \cup e^4 \cup S^3$ which actually maps S^5 into $e^{*4} \cup S^3$. However, such a map is by hypothesis homotopic to a map of S^5 into S^3 and therefore corresponds to the zero of

$$\pi_5(e^{*4} \cup e^4 \cup S^3) - i_5 \pi_5(e^4 \cup S^3).$$

This contradiction establishes the first part of the lemma.

Let $g_5\colon \pi_5(E^4, \dot{E}^4) \to \pi_5(e^4 \cup S^3, S^3)$ be induced by the characteristic map for e^4, and let β_0, β be the homotopy boundary homomorphisms $\beta_0\colon \pi_5(E^4, \dot{E}^4) \to \pi_4(\dot{E}^4)$, $\beta\colon \pi_5(e^4 \cup S^3, S^3) \to \pi_4(S^3)$, and consider the diagram

$$\begin{tikzcd} & & \pi_5(E^4, \dot{E}^4) \arrow[r, "\beta_0"] \arrow[d, "g_5"] & \pi_4(\dot{E}^4) \arrow[d, "h_4"] & \\ \pi_5(S^3) \arrow[r, "i_5"] & \pi_5(e^4 \cup S^3) \arrow[r, "j_5"] & \pi_5(e^4 \cup S^3, S^3) \arrow[r, "\beta"] & \pi_4(S^3) \arrow[r, "i_4"] & \pi_4(e^4 \cup S^3). \end{tikzcd}$$

Now $\pi_5(e^4 \cup S^3) - i_5 \pi_5(S^3) \approx \beta^{-1}(0)$, and the homomorphism of $h_4^{-1}(0)$ onto $\pi_5(e^4 \cup S^3) - i_5 \pi_5(S^3)$ is effectively† $g_5 \beta_0^{-1} \mid h_4^{-1}(0)\colon h_4^{-1}(0) \to \beta^{-1}(0)$. We have proved that this homomorphism is an isomorphism and, moreover, $h_4^{-1}(0) = \pi_4(\dot{E}^4)$. Thus $g_5 \beta_0^{-1}$ is an isomorphism, i.e. g_5 is an isomorphism. That g_5 is onto follows either from Theorem 1 of (**5**) or by observing that g_5 is onto $\beta^{-1}(0)$ and, since i_4 is an isomorphism,

$$\beta^{-1}(0) = \pi_5(e^4 \cup S^3, S^3).$$

The remainder of the proof now follows exactly the same course as the proof of the last two parts of Lemma 3.6.

To complete the apparatus for calculating $\pi_5(K^4)$ we require a lemma whose role is analogous to that of Lemma 2.1 in the calculation of $\pi_{n+2}(K^{n+1})$ $(n > 3)$.

† This statement follows without difficulty from the definition of the homomorphism ψ in Theorem 8 of (**3**). See also Theorem 2 of (**5**).

Lemma 5.5. *Let C be the union of N subcomplexes $\bar{e}_i^4$ $(i = 1,..., N)$ with a single common point, and let each subcomplex be formed by attaching to S_i^3 a 4-cell by a map of degree d_i. Let $d_{ij} = (d_i, d_j)$, the H.C.F. of d_i and d_j. Then*†

$$\pi_5(C) \approx \pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_N^4)+\sum_{i<j\leqslant N}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}.$$

We prove this by induction on N. Let us write C_N for

$$\bar{e}_1^4 \cup ... \cup \bar{e}_{N-1}^4 \cup S_N^3.$$

Then, by (5.1),

$$\pi_5(C_N) \approx \pi_5(\bar{e}_1^4 \cup ... \cup \bar{e}_{N-1}^4)+\pi_5(S_N^3)+[\pi_3(\bar{e}_1^4 \cup ... \cup \bar{e}_{N-1}^4), \pi_3(S_N^3)].$$

By the inductive hypothesis,

$$\pi_5(\bar{e}_1^4 \cup ... \cup \bar{e}_{N-1}^4) \approx \pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_{N-1}^4)+\sum_{i<j\leqslant N-1}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}.$$

Also

$$[\pi_3(\bar{e}_i^4), \pi_3(S_N^3)] = ([\pi_3(S_i^3), \pi_3(S_N^3)])_{d_i} \quad (i = 1,..., N-1).$$

To prove this last relation, it is obviously only necessary to take $i = 1$, $N = 2$. Then $\pi_5(S_1^3 \cup S_2^3) \approx \pi_5(S_1^3)+\pi_5(S_2^3)+[\pi_3(S_1^3), \pi_3(S_2^3)]$, and, by (5.1), $\pi_5(\bar{e}_1^4 \cup S_2^3) \approx \pi_5(\bar{e}_1^4)+\pi_5(S_2^3)+[\pi_3(\bar{e}_1^4), \pi_3(S_2^3)]$. Since any element in $\pi_3(\bar{e}_1^4)$ has a representative map $S^3 \to S_1^3$, it is clear that, under the injection i_5: $\pi_5(S_1^3 \cup S_2^3) \to \pi_5(\bar{e}_1^4 \cup S_2^3)$, $[\pi_3(S_1^3), \pi_3(S_2^3)]$ is mapped onto $[\pi_3(\bar{e}_1^4), \pi_3(S_2^3)]$. The kernel of $i_5 \mid [\pi_3(S_1^3), \pi_3(S_2^3)]$ is $i_5^{-1}(0) \cap [\pi_3(S_1^3), \pi_3(S_2^3)]$ and this group is, by (5.2), $d_1[\pi_3(S_1^3), \pi_3(S_2^3)]$. Thus

$$([\pi_3(S_1^3), \pi_3(S_2^3)])_{d_1} = [\pi_3(\bar{e}_1^4), \pi_3(S_2^3)].$$

We have shown that

$$\pi_5(C_N) \approx \pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_{N-1}^4)+\pi_5(S_N^3)+ \\ +\sum_{i<j\leqslant N-1}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}+\sum_{i=1}^{N-1}([\pi_3(S_i^3), \pi_3(S_N^3)])_{d_i}.$$

Now let d_N be odd. Then it follows as in the proof of Lemma 5.3 that $\pi_5(C) = i_5\pi_5(C_N)$. By (5.2), $i_5^{-1}(0) = \{h_5\pi_5(\dot{E}_N^4), [h_3\pi_3(\dot{E}_N^4), \pi_3(C_N)]\}$, where h_r: $\pi_r(\dot{E}_N^4) \to \pi_r(C_N)$ is, as usual, the homomorphism induced by the attaching map for e_N^4. Thus $i_5^{-1}(0) = \pi_5(S_N^3)+\sum_{i=1}^{N-1} d_N[\pi_3(S_N^3), \pi_3(\bar{e}_i^4)]$. Since $(\{[\pi_3(S_N^3), \pi_3(S_i^3)]\}_{d_i})_{d_N} = ([\pi_3(S_N^3), \pi_3(S_i^3)])_{d_{iN}}$, it follows that

$$\pi_5(C) \approx \pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_{N-1}^4)+\sum_{i<j\leqslant N}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}.$$

Since $\pi_5(\bar{e}_N^4) = 0$, this establishes the induction if d_N is odd.

† If G is an abelian group and σ a positive integer, $(G)_\sigma$ stands for the difference group $G-\sigma G$. Here $[\pi_3(S_i^3), \pi_3(S_j^3)]$ is a cyclic infinite group generated by the class of the 'Whitehead product' map $S^5 \to S_i^3 \cup S_j^3$, so that $([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}$ is cyclic of order d_{ij}.

Now assume that d_N is even. Then it follows as in the proof of Lemma 5.4 that there is a homomorphism of $\pi_4(\dot{E}_N^4)$ onto $\pi_5(C)-i_5\pi_5(C_N)$. Moreover this homomorphism is an isomorphism. For consider the diagram

$$\begin{array}{ccc} \pi_5(C_N) & \xrightarrow{i_5} & \pi_5(C) \\ \downarrow\lambda & & \downarrow\mu \\ \pi_5(S_N^3) & \xrightarrow{i_5'} & \pi_5(\bar{e}_N^4), \end{array}$$

where λ, μ are induced by the projections $C_N \to S_N^3$, $C \to \bar{e}_N^4$. Then λ, μ are homomorphisms onto and $\mu i_5 = i_5'\lambda$. If i_5 were onto, it would follow that i_5' was onto, and we know by Lemma 5.4 that this is not so. It is, in fact, clear that a map of S^5 into $\bar{e}_N^4$ which is not homotopic, in $\bar{e}_N^4$, to a map of S^5 into S_N^3, does not belong to a class in $i_5\pi_5(C_N)$. Now, since d_N is even, it follows from (5.2) that

$$i_5^{-1}(0) = \sum_{i=1}^{N-1} d_N[\pi_3(S_N^3), \pi_3(\bar{e}_i^4)].$$

Thus

$$\pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_{N-1}^4)+\pi_5(S_N^3)+\sum_{i<j\leqslant N}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}$$

is isomorphically embedded in $\pi_5(C)$ as a subgroup of index 2. We have shown that any element in $\pi_5(C)$ may be expressed as $a+b$, with

$$a \in \pi_5(\bar{e}_1^4)+...+\pi_5(\bar{e}_{N-1}^4)+\sum_{i<j\leqslant N}([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}, \quad b \in \pi_5(\bar{e}_N^4).$$

Now let $\lambda_1+...+\lambda_N+\sum_{i<j\leqslant N}\mu_{ij} = 0$, where $\lambda_k \in \pi_5(\bar{e}_k^4)$, $k = 1,..., N$, and $\mu_{ij} \in ([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_{ij}}$. Now $\lambda_1+...+\lambda_{N-1}+\sum_{i<j\leqslant N}\mu_{ij} \in i_5\pi_5(C_N)$, so that $\lambda_N \in \pi_5(S_N^3)$. However, $i_5\pi_5(C_N)$ does break up into direct summands, so that $\lambda_1 = ... = \lambda_N = \mu_{12} = ... = \mu_{N-1,N} = 0$. This establishes the induction if d_N is even, and completes the proof of the lemma.

I recall that in the course of the proof of this lemma we have shown that

$$[\pi_3(S_i^3), \pi_3(\bar{e}_j^4)] = ([\pi_3(S_i^3), \pi_3(S_j^3)])_{d_j}. \tag{5.6}$$

For simplicity, and without loss of generality, let us assume that $\sigma_1,..., \sigma_t$ are torsion coefficients, so that $\sigma_1|\sigma_2|...|\sigma_t$ and $(\sigma_i, \sigma_j) = \sigma_i$ if $i < j$. It then follows from Lemmas 5.3, 5.4, 5.5, and (5.1), (5.6), that†

$$\pi_5(K^4) \approx \sum_{i<j\leqslant m} R_{ij}+P_3(\sigma_{h+1})+...+P_3(\sigma_t)+(\bar{a}_{l+1},..., \bar{a}_m)+(\bar{b}_1^*,..., \bar{b}_l^*), \tag{5.7}$$

where R_{ij} is cyclic of order σ_i if $i \leqslant t$ and is cyclic infinite otherwise.

† See (3.7).

The consequences of attaching the 5-cells are exactly those of attaching the $(n+2)$-cells, when $n > 3$. We have thus proved

THEOREM 5.8. *Let K be an A_3^2-polyhedron and let $H_3(K)$ have Betti number $m-t$ and torsion coefficients $\sigma_1,\ldots, \sigma_t$, with $\sigma_i \mid \sigma_j$ $(i < j)$. Then the fifth homotopy group of K is given in terms of its homology system by the isomorphism*

$$\pi_5(K) \approx \sum_{i<j\leqslant m} R_{ij} + \mu H_4 + H_5 + E,$$

where E is an extension of ${}_2H_3$ by $H_3(2) - \gamma H_5(2)$, and R_{ij} is cyclic of order σ_i if $i \leqslant t$ and cyclic infinite otherwise.

COROLLARY 5.9. *If, in addition, K has no 3-dimensional torsion coefficients of the form $4r+2$, then*

$$\pi_5(K) \approx \sum_{i<j\leqslant m} R_{ij} + (H_3(2) - \gamma H_5(2)) + {}_2H_3 + \mu H_4 + H_5.$$

The discussion of the nature of the extension E follows exactly the same lines as in § 3. We also have available the analogue of Theorem 4.1 in the form

THEOREM 5.10. *Let K be an A_3^2-polyhedron such that $\pi_4(K) = 0$, and let $H_3(K)$ have Betti number $m-t$ and torsion coefficients $\sigma_1,\ldots, \sigma_t$, with $\sigma_i \mid \sigma_j$ $(i < j)$. Then*

$$\pi_5(K) \approx \sum_{i<j\leqslant m} R_{ij} + {}_2H_3 + H_5,$$

where R_{ij} is cyclic of order σ_i if $i \leqslant t$, and cyclic infinite otherwise.

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